# Triple-layer Tissue Prediction for Cutaneous Skin Burn Injury: Analytical and Parametric Analysis

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## Abstract

This paper demonstrates a non-Fourier prediction methodology of triple-layer human skin tissue for determining skin burn injury with non-ideal properties of tissue, metabolism and blood perfusion. The dual-phase lag (DPL) bioheat model is employed and solved using joint integral transform (JIT) through Laplace and Fourier transform method. Parametric studies on the effects of skin tissue properties, initial temperature, blood perfusion rate and heat transfer parameters for the thermal response and exposure time of the layers of the skin tissue are carried out. The study demonstrates that the initial tissue temperature, the thermal conductivity of the epidermis and dermis, relaxation time, thermalisation time and convective heat transfer coefficient are critical parameters to examine skin burn injury threshold. The study also shows that thermal conductivity and the blood perfusion rate exhibits negligible effects on the burn injury threshold. The objective of the present study is to support the accurate quantification and assessment of skin burn injury for reliable experimentation, design and optimisation of thermal therapy delivery.

**Keywords:** Analytical method; Bioheat model; Burns; Dual-phase lag model; Laplace-Fourier integral method

## 1. Introduction

Burn is a critical cause of mortality and morbidity globally. Burns from thermal, electrical, and chemical mechanisms occur mainly in the home, industrial environments, and extreme situations such as in military combat. Globally, an average of 180,000 burn-related deaths is reported yearly, with burns treatment been a budget concern and among the leading causes of disability-adjusted life years [1, 2]. For example, an estimated 250,000 individuals each year experience burn-related injuries in the UK, where about 175,000 individual visits the A&E Departments with burn injuries and 13,000 of these individuals are admitted to the hospital where 6,400 are children with scald being the prevalent burn injury. Consequently, the economic cost of burns treatment and its associated damage is huge. In a recent report, the cost of burn treatments and services by the NHS is calculated at more than £20 million per annum [3]. Burn victims often require long periods of rehabilitation and could receive multiple skin grafts and painful physical treatment, often resulting in lifelong psychological and physical scars [4]. Therefore, to support the timely clinical and appropriate management of burns, different studies have been carried in the literature to model heat transfer processes in the living tissues.

Heat transport in living tissue is a complex process and developing its thermal models is often a challenging task due to several biological ad thermal processes including conductive heat transfer, convective heat transfer, radiative heat transfer, metabolism, evaporation and phase change. Heat transport in the skin is achieved through conductive heat transfer processes including metabolic heat generation, blood circulation, sweating, and sometimes heat dissipation through the air above the skin surface. The subject of heat transport in living tissue was first experimentally investigated by Pennes [5]. The Pennes bioheat model presents a suitable modification of the standard heat equation by introducing a blood perfusion term. Moreover, the Pennes bioheat model is widely employed due to its simplicity and validity in different biomedical simulations including ablation of tissues using lasers, laser surgical processes, thermal diagnostics, and thermal parameter estimation. However, the Pennes bioheat model is based on the classical Fourier's law on the assumption that thermal propagation is infinite which in physical reality is impractical. Also, living tissues are highly non-homogeneous and require relaxation time to accumulate enough energy to transfer to the nearest element. The limitations of the Pennes bioheat model were independently addressed by Cattaneo [6] and Vernotte [7] using the non-Fourier model to consider the finite propagation speed of heat. The resulting hyperbolic or thermal wave model due to the wave-like characteristics of heat transport constitutes the Cattaneo and Vernotte constitutive relationship. To this end, several studies on heat transfer in skin tissue employ the thermal wave bioheat model due to its established experimental validations and ability to produce a more accurate prediction than the classical Fourier model [8-10]. The thermal wave model only treats micro-scale response in time and not the micro-scale response in space [11-13]. However, in [11], Tzou proposed a new model that considered the micro-scale response in space using phase lag in heat flux and temperature gradient. The phase lag model is effective for freezing and thawing processes with phase change and is effective in various biomedical applications such as cryosurgery and cryopreservation to prevent biological materials including cells, tissues, organs [14-16]. To examine the effects of microstructural interactions in the fast transient process of heat transport without phase change, the DPL bioheat model has been widely used to study different thermal therapeutic treatments including hyperthermia treatment, thermal diagnostic and comfort analysis and burn injury evaluation of skin tissue [17-23]. The DPL model is based on the first-order Taylor series expansion. However, for accurate prediction of temperature distribution in biological tissues, which is critical in various therapeutic burn treatments, a rational predictive model becomes requisite. Additionally, in biological tissues phase change occurs over a wide range and there exist moving boundaries between two phases which result in nonlinear mathematical models. Further, due to difference in biological and thermal properties of the skin, the computation of boundary conditions at the interface between two adjacent layers is often a complex task. Consequently, different studies in the literature employ the use of multi-dimensional bioheat models for single, double and triple layer thermal assessments [24-29]. Therefore, the present study focus on the modified DPL bioheat transfer model based on second-order Taylor expansion to demonstrate the non-Fourier thermal modelling of triple-layer cutaneous tissue for prediction of skin burn with non-ideal properties of tissue, metabolism and blood perfusion. The developed DPL bioheat models are solved analytically using JIT through Laplace and Fourier transform methods. Parametric studies are carried out and the obtained results are presented. The obtained results have been convincingly validated with other methods of the published literature. The results obtained from the present analysis using JIT are proposed to help in the quantification of skin burns, which will assist to develop novel treatments and pain relief.

## 2. Model of Human Skin Exposed to Burn

The human skin helps protect the body from microbes and the elements as well as support regulating body temperature under various thermal conditions. The human skin is constructed of two distinct layers: epidermis and dermis, with the hypodermis or subcutaneous layer, which is not part of the skin, but included in the skin structure, since it attaches to the dermis by collagen and elastic fibres as illustrated in Fig. 1.

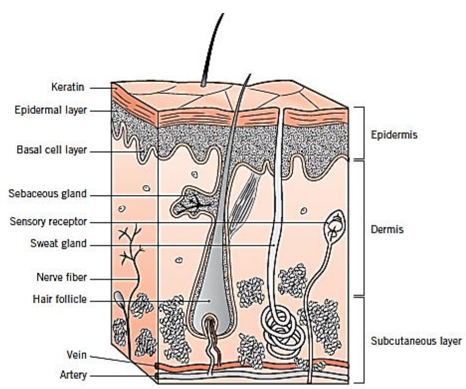


Fig. 1. Morphology of human skin

The thickness of each layer depends on the skin position on the human body. The epidermis consists of living and dead cells and comprises of 95% keratinocytes and 5% non-keratinocytes, respectively. The dermis made up of little reticulin, collagen, elastin, and group substances which are fibrin proteins play the vital role of thermoregulation and support the vascular network to supply the non-vascularised epidermis with nutrients. Moreover, the subcutaneous layer contains the loose fatty connective tissues and carries blood vessels and nerves to the overlying skin [30]. To the deal with the paradox of the classical Fourier's model and account for the limitations in the thermal wave model, the effect of microstructural interactions in the fast-transient process of heat transport is considered. The consideration is based on the fact that the gradient of temperature at a point in the material at time $t + \tau_T$ corresponds to the heat flux vector at the same point at time $t + \tau_q$, which can be written mathematically as:

$$q\left(x, t+\tau_q\right) = -k\nabla T\left(x, t+\tau_T\right) \tag{1}$$

where $\tau_q$ and $\tau_T$ are non-zero times and accounts for the effects of thermal inertia and microstructural interactions. $\tau_q$ is the phase-lag to establish heat flux and its associated conduction through a medium, $\tau_T$ accounts for the diffusion of induced heat by $\tau_q$, and represents the phase-lag to establish the temperature gradient across the medium when conduction occurs through the small-scale structures.

Using the second-order Taylor expansions, the DPL model can be expressed as:

$$\left(1+\tau_q\frac{\partial}{\partial t}+\frac{\tau_q^2}{2}\frac{\partial^2}{\partial t^2}\right)\vec{q}=-\left(1+\tau_T\frac{\partial}{\partial t}+\frac{\tau_T^2}{2}\frac{\partial^2}{\partial t^2}\right)k\nabla T \tag{2}$$

The above model covers a wide range of space and time for physical observations.

From Eq. (2), we write

$$\left(1+\tau_q\frac{\partial}{\partial t}+\frac{\tau_q^2}{2}\frac{\partial^2}{\partial t^2}\right)\nabla\vec{q}=-\left(1+\tau_T\frac{\partial}{\partial t}+\frac{\tau_T^2}{2}\frac{\partial^2}{\partial t^2}\right)(\nabla k\nabla T) \tag{3}$$

From the local energy balance, the equation for conservation of energy can be written as:

$$\rho c_p\frac{\partial T}{\partial t}=-\nabla\cdot\vec{q}+\omega_b c_b(T_b-T)+q_m+q_{ext}+q_{evap} \tag{4}$$

Eq. (4) can be re-arranged as:

$$-\nabla\cdot\vec{q}=\rho c_p\frac{\partial T}{\partial t}-\omega_b c_b(T_b-T)-q_m-q_{ext}-q_{evap} \tag{5}$$

by substituting Eq. (5) in Eq. (3) we arrived at

$$\begin{aligned}&\left(1+\tau_q\frac{\tau_q^2}{\partial t}+\frac{\partial}{2}\frac{\partial^2}{\partial t^2}\right)\left(\rho c_p\frac{\partial T}{\partial t}-\omega_b c_b\left(T_b-T\right)-q_m-q_{ext}-q_{evap}\right)\\&=\left(1+\tau_T\frac{\tau_T^2}{\partial t}+\frac{\partial}{2}\frac{\partial^2}{\partial t^2}\right)(\nabla.k\nabla T)\end{aligned} \tag{6}$$

The expansion of Eq. (6) produces

$$\begin{aligned}&\rho c_p\frac{\partial T}{\partial t}+\tau_q\rho c_p\frac{\partial^2 T}{\partial t^2}+\frac{\tau_q^2}{2}\rho c_p\frac{\partial^3 T}{\partial t^3}+\omega_b c_b\tau_b\frac{\partial T}{\partial t}+\omega_b c_b\frac{\tau_q^2}{2}\frac{\partial^2 T}{\partial t^2}\\&=(\nabla.k\nabla T)+\tau_T\frac{\partial}{\partial t}(\nabla.k\nabla T)+\frac{\tau_T^2}{2}\frac{\partial^2}{\partial t^2}(\nabla.k\nabla T)+\omega_b c_b\left(T_b-T\right)\\&+q_m+\tau_q\frac{\partial q_m}{\partial t^2}+\frac{\tau_q^2}{2}\frac{\partial^2 q_m}{\partial t^2}+q_{ext}+\tau_q\frac{\partial q_{ext}}{\partial t}+\frac{\tau_q^2}{2}\frac{\partial^2 q_{ext}}{\partial t^2}-q_{evap}-\tau_q\frac{\partial q_{evap}}{\partial t}-\frac{\tau_q^2}{2}\frac{\partial^2 q_{evap}}{\partial t^2}\end{aligned} \tag{7}$$

From the above DPL model in Eq. (7), if $\tau_q$ and $\tau_T$ are set to zero, then the Pennes' model is recovered.

$$\rho c_p\frac{\partial T}{\partial t}=(\nabla.k\nabla T)+\omega_b c_b\left(T_b-T\right)+q_m+q_{ext} \tag{8}$$

However, if $\tau_T$ is set to zero and the first-order Taylor expansion is used only in time, then the thermal wave or hyperbolic model is recovered as:

$$\begin{aligned}&\rho c_p\frac{\partial T}{\partial t}+\tau_q\rho c_p\frac{\partial^2 T}{\partial t^2}+\omega_b c_b\tau_q\frac{\partial T}{\partial t}=(\nabla.k\nabla T)+\omega_b c_b\left(T_b-T\right)\\&+q_m+\tau_q\frac{\partial q_m}{\partial t}+q_{ext}+\tau_q\frac{\partial q_{ext}}{\partial t}-q_{evap}-\tau_q\frac{\partial q_{evap}}{\partial t}\end{aligned} \tag{9}$$

Further, if $\tau_T$ is set to zero and the first-order Taylor expansion is used both in time and space, then Tzou' model in [11] is recovered as:

$$\rho c_p \frac{\partial T}{\partial t} + \tau_q \rho c_p \frac{\partial^2 T}{\partial t^2} + \omega_b c_b \tau_q \frac{\partial T}{\partial t} = (\nabla . k \nabla T) + \tau_T \frac{\partial}{\partial t}(\nabla . k \nabla T)$$
$$+\omega_b c_b (T_b - T) + q_m + \tau_q \frac{\partial q_m}{\partial t} + q_{ext} + \tau_q \frac{\partial q_{ext}}{\partial t} - q_{evap} - \tau_q \frac{\partial q_{evap}}{\partial t} \tag{10}$$

The three-dimensional DPL model in the Cartesian coordinates gives

$$\rho c_p \frac{\partial T}{\partial t} + \tau_q \rho c_p \frac{\partial^2 T}{\partial t^2} + \frac{\tau_q^2}{2} \rho c_p \frac{\partial^3 T}{\partial t^3} + +\omega_b c_b \tau_q \frac{\partial T}{\partial t} + \omega_b c_b \frac{\tau_q^2}{2} \frac{\partial^2 T}{\partial t^2}$$
$$= \frac{\partial}{\partial x}\left(k \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k \frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k \frac{\partial T}{\partial z}\right)$$
$$+\tau_T \frac{\partial}{\partial t}\left(\frac{\partial}{\partial x}\left(k \frac{\partial T}{\partial x}\right)\right) + \tau_T \frac{\partial}{\partial t}\left(\frac{\partial}{\partial y}\left(k \frac{\partial T}{\partial y}\right)\right) + \tau_T \frac{\partial}{\partial t}\left(\frac{\partial}{\partial z}\left(k \frac{\partial T}{\partial z}\right)\right) \tag{11}$$
$$+\frac{\tau_T^2}{2} \frac{\partial^2}{\partial t^2}\left(\frac{\partial}{\partial x}\left(k \frac{\partial T}{\partial x}\right)\right) + \frac{\tau_T^2}{2} \frac{\partial^2}{\partial t^2}\left(\frac{\partial}{\partial y}\left(k \frac{\partial T}{\partial y}\right)\right) + \frac{\tau_T^2}{2} \frac{\partial^2}{\partial t^2}\left(\frac{\partial}{\partial z}\left(k \frac{\partial T}{\partial z}\right)\right)$$
$$+\omega_b c_b (T_b - T) + q_m + \tau_q \frac{\partial q_m}{\partial t} + \frac{\tau_q^2}{2} \frac{\partial^2 q_m}{\partial t^2} + q_{ext} + \tau_q \frac{\partial q_{ext}}{\partial t} + \frac{\tau_q^2}{2} \frac{\partial^2 q_{ext}}{\partial t^2}$$
$$-q_{evap} + \tau_q \frac{\partial q_{evap}}{\partial t} + \frac{\tau_q^2}{2} \frac{\partial^2 q_{evap}}{\partial t^2}$$

by taking the thermal conductivity as constant, the three-dimensional DPL model in the Cartesian coordinate gives

$$\rho c_p \frac{\partial T}{\partial t} + \tau_q \rho c_p \frac{\partial^2 T}{\partial t^2} + \frac{\tau_q^2}{2} \rho c_p \frac{\partial^3 T}{\partial t^3} + +\omega_b c_b \tau_q \frac{\partial T}{\partial t} + \omega_b c_b \frac{\tau_q^2}{2} \frac{\partial^2 T}{\partial t^2}$$
$$= k\left(\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) + \tau_T \frac{\partial}{\partial t}\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) + \frac{\tau_T^2}{2} \frac{\partial^2 T}{\partial t^2}\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right)\right) \tag{12}$$
$$\omega_b c_b (T_b - T) + q_m + \tau_q \frac{\partial q_m}{\partial t} + \frac{\tau_q^2}{2} \frac{\partial^2 q_m}{\partial t^2} + q_{ext} + \tau_q \frac{\partial q_{ext}}{\partial t} + \frac{\tau_q^2}{2} \frac{\partial^2 q_{ext}}{\partial t^2} - q_{evap} - \tau_q \frac{\partial q_{evap}}{\partial t} - \frac{\tau_q^2}{2} \frac{\partial^2 q_{evap}}{\partial t^2}$$

The two-dimensional DPL model in the Cartesian coordinates gives

$$\rho c_p \frac{\partial T}{\partial t} + \tau_q \rho c_p \frac{\partial^2 T}{\partial t^2} + \frac{\tau_q^2}{2} \rho c_p \frac{\partial^3 T}{\partial t^3} + +\omega_b c_b \tau_q \frac{\partial T}{\partial t} + \omega_b c_b \frac{\tau_q^2}{2} \frac{\partial^2 T}{\partial t^2}$$
$$= \frac{\partial}{\partial x}\left(k \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial z}\left(k \frac{\partial T}{\partial z}\right) + \tau_T \frac{\partial}{\partial t}\left(\frac{\partial}{\partial x}\left(k \frac{\partial T}{\partial x}\right)\right) + \tau_T \frac{\partial}{\partial t}\left(\frac{\partial}{\partial z}\left(k \frac{\partial T}{\partial z}\right)\right) \tag{13}$$
$$+\frac{\tau_T^2}{2} \frac{\partial^2 T}{\partial t^2}\left(\frac{\partial}{\partial x}\left(k \frac{\partial T}{\partial x}\right)\right) + \frac{\tau_T^2}{2} \frac{\partial^2}{\partial t^2}\left(\frac{\partial}{\partial z}\left(k \frac{\partial T}{\partial z}\right)\right) + \omega_b c_b (T_b - T)$$
$$+q_m + \tau_q \frac{\partial q_m}{\partial t} + \frac{\tau_q^2}{2} \frac{\partial^2 q_m}{\partial t^2} + q_{ext} + \tau_q \frac{\partial q_{ext}}{\partial t} + \frac{\tau_q^2}{2} \frac{\partial^2 q_{ext}}{\partial t^2} - q_{evap} - \tau_q \frac{\partial q_{evap}}{\partial t} - \frac{\tau_q^2}{2} \frac{\partial^2 q_{evap}}{\partial t^2}$$

Taking the thermal conductivity as constant, the three-dimensional DPL model in the Cartesians co-ordinates produces

$$\rho c_p \frac{\partial T}{\partial t} + \tau_q \rho c_p \frac{\partial^2 T}{\partial t^2} + \frac{\tau_q^2}{2} \rho c_p \frac{\partial^3 T}{\partial t^3} + +\omega_b c_b \tau_q \frac{\partial T}{\partial t} + \omega_b c_b \frac{\tau_q^2}{2} \frac{\partial^2 T}{\partial t^2}$$
$$= k\left(\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) + \tau_T \frac{\partial}{\partial x}\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) + \frac{\tau_T^2}{2} \frac{\partial^2}{\partial t^2}\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right)\right) \tag{14}$$
$$+\omega_b c_b (T_b - T) + q_m + \tau_q \frac{\partial q_m}{\partial t} + \frac{\tau_q^2}{2} \frac{\partial^2 q_m}{\partial t^2} + q_{ext} + \tau_q \frac{\partial q_{ext}}{\partial t} + \frac{\tau_q^2}{2} \frac{\partial^2 q_{ext}}{\partial t^2}$$
$$-q_{evap} - \tau_q \frac{\partial q_{evap}}{\partial t} - \frac{\tau_q^2}{2} \frac{\partial^2 q_{evap}}{\partial t^2}$$

The one-dimensional DPL model with varying thermal conductivity is expressed as

$$\rho c_p \frac{\partial T}{\partial t} + \tau_q \rho c_p \frac{\partial^2 T}{\partial t^2} + \frac{\tau_q^2}{2} \rho c_p \frac{\partial^3 T}{\partial t^3} + +\omega_b c_b \tau_q \frac{\partial T}{\partial t} + \omega_b c_b \frac{\tau_q^2}{2} \frac{\partial^2 T}{\partial t^2}$$
$$= \frac{\partial}{\partial x}\left(k \frac{\partial T}{\partial x}\right) + \tau_T \frac{\partial}{\partial t}\left(\frac{\partial}{\partial x}\left(k \frac{\partial T}{\partial x}\right)\right) + \frac{\tau_T^2}{2} \frac{\partial^2}{\partial t^2}\left(\frac{\partial}{\partial x}\left(k \frac{\partial T}{\partial x}\right)\right) \tag{15}$$
$$+\omega_b c_b (T_b - T) + q_m + \tau_q \frac{\partial q_m}{\partial t} + \frac{\tau_q^2}{2} \frac{\partial^2 q_m}{\partial t^2} + q_{ext} + \tau_q \frac{\partial q_{ext}}{\partial t} + \frac{\tau_q^2}{2} \frac{\partial^2 q_{ext}}{\partial t^2}$$
$$-q_{evap} - \tau_q \frac{\partial q_{evap}}{\partial t} - \frac{\tau_q^2}{2} \frac{\partial^2 q_{evap}}{\partial t^2}$$

Then the one-dimensional DPL model with constant thermal conductivity is expressed as:

$$\rho c_p \frac{\partial T}{\partial t} + \tau_q \rho c_p \frac{\partial^2 T}{\partial t^2} + \frac{\tau_q^2}{2} \rho c_p \frac{\partial^3 T}{\partial t^3} + +\omega_b c_b \tau_q \frac{\partial T}{\partial t} + \omega_b c_b \frac{\tau_q^2}{2} \frac{\partial^2 T}{\partial t^2}$$
$$= k\left(\frac{\partial^2 T}{\partial x^2} + \tau_T \frac{\partial}{\partial t}\left(\frac{\partial^2 T}{\partial x^2}\right) + \frac{\tau_T^2}{2} \frac{\partial^2}{\partial t^2}\left(\frac{\partial^2 T}{\partial x^2}\right)\right) + \omega_b c_b (T_b - T) \tag{16}$$
$$+q_m + \tau_q \frac{\partial q_m}{\partial t} + \frac{\tau_q^2}{2} \frac{\partial^2 q_m}{\partial t^2} + q_{ext} + \tau_q \frac{\partial q_{ext}}{\partial t} + \frac{\tau_q^2}{2} \frac{\partial^2 q_{ext}}{\partial t^2}$$
$$-q_{evap} - \tau_q \frac{\partial q_{evap}}{\partial t} - \frac{\tau_q^2}{2} \frac{\partial^2 q_{evap}}{\partial t^2}$$

## 2.1. Modelling Evaporation during the High Heating

During heating of tissues to high temperature, the skin tissue moisture, normally around 70-75% evaporates, absorbing latent heat. This is because at 100°C, during ablation, water vaporisation in the tissues occurs leading to dehydration of the tissues. As the temperature increases (>100°C), the continuous vaporisation and dehydration of the tissue initiate the carbonisation process of the tissues [31-33]. These biological processes are fundamental in the analysis of skin burn at high temperature and without due consideration of these processes, the results from the models would differ significantly from experimental results. To this end, the vaporization terms are included in the above models to accommodate the phase change due to evaporation. Under such condition, the enthalpy-based model can be adopted [34]. In such a model, the enthalpy contains a three-part description. The first part corresponds to temperature change in the liquid-containing tissue, the second part accounts for the latent heat of evaporation and the third part deals with the temperature change in the post-phase-change tissue. Therefore, without loss of generality, in this work, a simple method to incorporate the simple water-related processes into the thermal models is employed to improve the ablation models at high temperatures. In the simple model, the rate of heat of vaporisation is modelled as established in [35] as

$$q_{evap} = -\lambda \frac{d\rho_w}{dt} \tag{17}$$

$\lambda$ and $\rho_w$ represents the latent heat of vaporisation for water (2260kJ/Kg) and tissue water density respectively.

The tissue water density is a function of temperature and Eq. (17) can be written using Chain rule as:

$$q_{evap} = -\lambda \frac{\partial \rho_w}{dt} = -\lambda \frac{\partial W}{\partial T} \cdot \frac{\partial T}{\partial t} \tag{18}$$

where $\frac{\partial W}{\partial T} < 0$ for evaporation

## 3. Two-dimensional DPL Model for the Triple Layer Skin Tissue

Consider the triple-layer skin tissue in Fig. 2, the two-dimensional triple-layer DPL thermal model can be written as

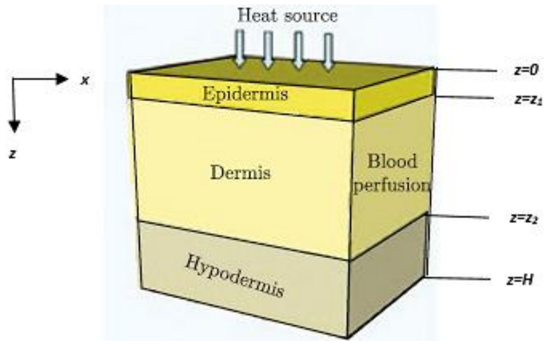


Fig. 2. Physical model of skin tissue with direct contact with the heat source

The two-dimensional, three-layer DPL thermal model can be written as:

$$\begin{aligned}
&\rho_l\left(c_{p,l}-\frac{\lambda_l}{\rho_l}\frac{\partial W_l}{\partial T}\right)\frac{\partial T_l}{\partial t}+\tau_{q,l}\rho_l\left(c_{p,l}-\frac{\lambda_l}{\rho_l}\frac{\partial W_l}{\partial T}\right)\frac{\partial^2 T_l}{\partial t^2}+\frac{\tau_{q,l}^2}{2}\rho_l\left(c_{p,l}-\frac{\lambda_l}{\rho_l}\frac{\partial W_l}{\partial T}\right)\frac{\partial^3 T_l}{\partial t^3}\\
&+\omega_b c_b\tau_{q,l}\frac{\partial T_l}{\partial t}+\omega_b c_b\frac{\tau_{q,l}^2}{2}\frac{\partial^2 T_l}{\partial t^2}=k_l\left(\left(\frac{\partial^2 T_l}{\partial x^2}+\frac{\partial^2 T_l}{\partial z^2}\right)+\tau_{T_l}\frac{\partial}{\partial t}\left(\frac{\partial^2 T_l}{\partial x^2}+\frac{\partial^2 T_l}{\partial z^2}\right)+\frac{\tau_{T_l}^2}{2}\frac{\partial^2}{\partial t^2}\left(\frac{\partial^2 T_l}{\partial x^2}+\frac{\partial^2 T_l}{\partial z^2}\right)\right)\quad for\ \ l=1,2,3.\\
&+\omega_b c_b\left(T_b-T_l\right)+q_m+\tau_{q,l}\frac{\partial q_m}{\partial t}+\frac{\tau_{q,l}^2}{2}\frac{\partial^2 q_m}{\partial t^2}+q_{ext}+\tau_{q,l}\frac{\partial q_{ext}}{\partial t}+\frac{\tau_{q,l}^2}{2}\frac{\partial^2 q_{ext}}{\partial t^2}
\end{aligned}\tag{19}$$

which can be written as

$$\begin{aligned}
&\rho_l c'_{p,l}\frac{\partial T_l}{\partial t}+\tau_{q,l}\rho_l c'_{p,l}\frac{\partial^2 T_l}{\partial t^2}+\frac{\tau_{q,l}^2}{2}\rho_l c'_{p,l}\frac{\partial^3 T_l}{\partial t^3}+\omega_b c_b\tau_{q,l}\frac{\partial T_l}{\partial t}+\omega_b c_b\frac{\tau_{q,l}^2}{2}\frac{\partial^2 T_l}{\partial t^2}\\
&=k_l\left(\left(\frac{\partial^2 T_l}{\partial x^2}+\frac{\partial^2 T_l}{\partial z^2}\right)+\tau_{T_l}\frac{\partial}{\partial t}\left(\frac{\partial^2 T_l}{\partial x^2}+\frac{\partial^2 T_l}{\partial z^2}\right)+\frac{\tau_{T_l}^2}{2}\frac{\partial^2}{\partial t^2}\left(\frac{\partial^2 T_l}{\partial x^2}+\frac{\partial^2 T_l}{\partial z^2}\right)\right)\quad for\ \ l=1,2,3.\\
&+\omega_b c_b\left(T_b-T_l\right)+q_m+\tau_{q,l}\frac{\partial q_m}{\partial t}+\frac{\tau_{q,l}^2}{2}\frac{\partial^2 q_m}{\partial t^2}+q_{ext}+\tau_{q,l}\frac{\partial q_{ext}}{\partial t}+\frac{\tau_{q,l}^2}{2}\frac{\partial^2 q_{ext}}{\partial t^2}
\end{aligned}\tag{20}$$

where

$c'_{p,l}=\left(c_{p,l}-\frac{\lambda_l}{\rho_l}\frac{\partial W_l}{\partial T}\right)$ is the effective heat capacity

The present study considers a skin insult scenario which consists of direct contact to the with a heat disk of 1cm diameter that is maintained at a defined constant temperature for a specified time. The surface area peripheral to the disc was assumed to experience convective heat transfer with an ambient air temperature of 25ºC during the burn process. After completion of the insult, the disc was removed and the entire skin surface was cooled by natural convection of the surrounding air [31, 32]. For the present scenario, the initial, boundary and interlayer conditions are stated as:

### 3.1.1. Initial conditions

At the initial condition, the temperature of the skin tissue is equal to the blood. Therefore,

$$t=0,\ T_l=T_{b,l},\ \frac{\partial T_l}{\partial t}=\frac{\partial^2 T_l}{\partial t^2}=0\qquad 0\le x\le L,\quad 0\le z\le H,\tag{21}$$

**3.1.2. The boundary conditions**,

$$t>0,\quad x=0,\quad 0\le z\le H,\quad \frac{\partial T_l}{\partial x}=0, \tag{22a}$$

$$t>0,\quad x=L,\quad 0\le z\le H,\quad \frac{\partial T_l}{\partial x}=0, \tag{22b}$$

where *l=1, 2*

$$t>0,\quad z=0,\quad 0\le x\le L\quad \frac{\partial T_1}{\partial z}=q \tag{22c}$$

$$t>0,\quad z=H,\quad 0\le x\le L,\quad \frac{\partial T_3}{\partial z}=0 \tag{22d}$$

**3.1.3. Interlayer conditions**

For the interlayers, the temperature and the heat flux in the respective layers must be equal at each point in between the layers as:

$$t>0,\quad z=z_l,\quad 0\le x\le L,\quad T_l=T_{l+1}, \tag{23a}$$

$$t>0,\quad z=z_l,\quad 0\le x\le L,\quad k_l\frac{\partial T_l}{\partial z}=k_{l+1}\frac{\partial T_{l+1}}{\partial z}, \tag{23b}$$

where *l=1, 2*

For accurate analysis, the DPL model for each layer of the skin tissue as illustrated in Fig. 2 is developed as:

**3.2.1 Layer 1: Epidermis Layer**

$$\begin{aligned}
&\rho_1 c'_{p,1}\frac{\partial T_1}{\partial t}+\tau_{q,1}\rho_1 c'_{p,1}\frac{\partial^2 T_l}{\partial t^2}+\frac{\tau_{q,1}^2}{2}\rho_1 c'_{p,1}\frac{\partial^3 T_1}{\partial t^3}+\omega_b c_b\tau_{q,1}\frac{\partial T_1}{\partial t}+\omega_b c_b\frac{\tau_{q,1}^2}{2}\frac{\partial^2 T_1}{\partial t^2}\\
&=k_1\left(\left(\frac{\partial^2 T_1}{\partial x^2}+\frac{\partial^2 T_1}{\partial z^2}\right)+\tau_{T_1}\frac{\partial}{\partial t}\left(\frac{\partial^2 T_1}{\partial x^2}+\frac{\partial^2 T_1}{\partial z^2}\right)+\frac{\tau_{T_1}^2}{2}\frac{\partial^2}{\partial t^2}\left(\frac{\partial^2 T_1}{\partial x^2}+\frac{\partial^2 T_1}{\partial z^2}\right)\right)\\
&+\omega_b c_b\left(T_b-T_1\right)+q_m+\tau_{q,1}\frac{\partial q_m}{\partial t}+\frac{\tau_{q,1}^2}{2}\frac{\partial^2 q_m}{\partial t^2}+q_{ext}+\tau_{q,1}\frac{\partial q_{ext}}{\partial t}+\frac{\tau_{q,1}^2}{2}\frac{\partial^2 q_{ext}}{\partial t^2}
\end{aligned} \tag{24}$$

**3.2.3 Layer 2: Dermis Layer**

$$\begin{aligned}
&\rho_2 c'_{p,2}\frac{\partial T_2}{\partial t}+\tau_{q,2}\rho_2 c'_{p,2}\frac{\partial^2 T_2}{\partial t^2}+\frac{\tau_{q,2}^2}{2}\rho_2 c'_{p,2}\frac{\partial^3 T_2}{\partial t^3}+\omega_b c_b\tau_{q,2}\frac{\partial T_2}{\partial t}+\omega_b c_b\frac{\tau_{q,2}^2}{2}\frac{\partial^2 T_2}{\partial t^2}\\
&=k_2\left(\left(\frac{\partial^2 T_2}{\partial x^2}+\frac{\partial^2 T_2}{\partial z^2}\right)+\tau_{T_2}\frac{\partial}{\partial t}\left(\frac{\partial^2 T_2}{\partial x^2}+\frac{\partial^2 T_2}{\partial z^2}\right)+\frac{\tau_{T_2}^2}{2}\frac{\partial^2}{\partial t^2}\left(\frac{\partial^2 T_2}{\partial x^2}+\frac{\partial^2 T_2}{\partial z^2}\right)\right)\\
&+\omega_b c_b\left(T_b-T_2\right)+q_m+\tau_{q,2}\frac{\partial q_m}{\partial t}+\frac{\tau_{q,2}^2}{2}\frac{\partial^2 q_m}{\partial t^2}+q_{ext}+\tau_{q,2}\frac{\partial q_{ext}}{\partial t}+\frac{\tau_{q,2}^2}{2}\frac{\partial^2 q_{ext}}{\partial t^2}
\end{aligned} \tag{25}$$

**3.2.3 Layer 3: Subcutaneous Layer**

$$
\begin{aligned}
&\rho_3 c'_{p,3}\frac{\partial T_3}{\partial t}+\tau_{q,3}\rho_3 c'_{p,3}\frac{\partial^2 T_3}{\partial t^2}+\frac{\tau_{q,3}^2}{2}\rho_3 c'_{p,3}\frac{\partial^3 T_3}{\partial t^3}+\omega_b c_b \tau_{q,3}\frac{\partial T_3}{\partial t}+\omega_b c_b\frac{\tau_{q,3}^2}{2}\frac{\partial^2 T_3}{\partial t^2}\\
&=k_2\left(\left(\frac{\partial^2 T_3}{\partial x^2}+\frac{\partial^2 T_3}{\partial z^2}\right)+\tau_{T3}\frac{\partial}{\partial t}\left(\frac{\partial^2 T_3}{\partial x^2}+\frac{\partial^2 T_3}{\partial z^2}\right)+\frac{\tau_{T_3}^2}{2}\frac{\partial^2}{\partial t^2}\left(\frac{\partial^2 T_3}{\partial x^2}+\frac{\partial^2 T_3}{\partial z^2}\right)\right)\\
&+\omega_b c_b\left(T_b-T_3\right)+q_m+\tau_{q,3}\frac{\partial q_m}{\partial t}+\frac{\tau_{q,3}^2}{2}\frac{\partial^2 q_m}{\partial t^2}+q_{ext}+\tau_{q,3}\frac{\partial q_{ext}}{\partial t}+\frac{\tau_{q,3}^2}{2}\frac{\partial^2 q_{ext}}{\partial t^2}
\end{aligned}
\tag{26}
$$

#### 3.3.1. Initial conditions

At the initial condition, the temperature of the skin tissue is equal to the blood and is expressed as:

$$t=0,\ 0\le x\le L,\ 0\le z\le H,\ T_1=T_b,\ \frac{\partial T_1}{\partial t}=\frac{\partial^2 T_1}{\partial t^2}=0\ , \tag{27a}$$

$$t=0,\ 0\le x\le L,\ 0\le z\le H,\ T_2=T_b,\ \frac{\partial T_2}{\partial t}=\frac{\partial^2 T_2}{\partial t^2}=0, \tag{27b}$$

$$t=0,\ 0\le x\le L,\ 0\le z\le H,\ T_3=T_b,\ \frac{\partial T_3}{\partial t}=\frac{\partial^2 T_3}{\partial t^2}=0, \tag{27c}$$

#### 3.3.2. The boundary conditions,

$$t>0,\quad x=0,\ 0\le z\le H,\ \frac{\partial T_1}{\partial x}=0, \tag{28a}$$

$$t>0,\quad x=0,\ 0\le z\le H,\ \frac{\partial T_2}{\partial x}=0, \tag{28b}$$

$$t>0,\quad x=0,\ 0\le z\le H,\ \frac{\partial T_3}{\partial x}=0, \tag{28c}$$

$$t>0,\quad x=L,\ 0\le z\le H,\ \frac{\partial T_1}{\partial x}=0, \tag{28d}$$

$$t>0,\quad x=L,\ 0\le z\le H,\ \frac{\partial T_2}{\partial x}=0, \tag{28e}$$

$$t>0,\quad x=L,\ 0\le z\le H,\ \frac{\partial T_3}{\partial x}=0, \tag{28f}$$

At the surface of the skin, the heat conducted to the surface is taken as the heat lost to ambient air through convection

$$t>0,\ \ \mathrm{z}=0,\ \ 0\le x\le H\quad -k\frac{\partial T_1}{\partial z}=h_{eff}\left(T_a-T_1\right) \tag{28g}$$

where the effective convective heat transfer coefficient is given as

$$h_{eff}=h+\sigma\varepsilon\left(T_a+T_1\right)\left(T_a^2+T_a^1\right) \tag{28h}$$

However, at the bottom of the subcutaneous layer, the local temperature is equal to the arterial temperature

$$t>0,\quad z=H,\ 0\le x\le L,\ T_3=T_b \tag{28i}$$

#### 3.3.3. Interlayer conditions

In the interlayers, temperature and heat flux is assumed continuous across the interface. Therefore, the temperature and the heat flux in the respective layers is equal at each point amongst the layers.

$$t>0,\ \ z=z_1,\quad 0\le x\le L,\quad T_1=T_2, \tag{29a}$$

$$t > 0,\ z = z_1,\quad 0 \le x \le L,\quad k_1 \frac{\partial T_1}{\partial z} = k_2 \frac{\partial T_2}{\partial z}, \tag{29b}$$

$$t > 0,\ z = z_2,\quad 0 \le x \le L,\quad T_2 = T_3, \tag{29c}$$

$$t > 0,\ z = z_2,\quad 0 \le x \le L,\quad k_2 \frac{\partial T_2}{\partial z} = k_3 \frac{\partial T_3}{\partial z}, \tag{29d}$$

## 4. Joint Integral Transform solutions of the DPL thermal model of the triple layer

The interlayer conditions make the analytical solution of the thermal models to be convoluted. Consequently, recourse is made to the hybrid Laplace-Fourier transform method is employed to solve the equation.

### 4.1.1. Laplace transform method

The Laplace transform method is applied over time in the transient DPL models. The Laplace transform of a real function $f(t)$ and its inversion formulas are defined as:

$$F(s) = \int_0^{\infty} e^{-st} f(t)dt \tag{30}$$

$$f(t) = \frac{1}{2\pi i} \int_{s-i\infty}^{s+i\infty} e^{-st} F(s)dt \tag{31}$$

where $s=a+ib$ $(a, b \in R)$ is a complex number

### 4.1.2. Generalised Finite Fourier transform method

The generalised finite Fourier transform of a real function *f(x)* and its inversion formulas are defined as:

$$F[f(x)] = \bar{f}(\beta_m) = \int_0^L K(\beta_m, x) f(x)dx \tag{32}$$

where $K$ is the kernel and the $\beta_m$ 's are the eigenvalues of the kernel. The associated eigenvalues depend on boundary conditions. The general inversion Finite Fourier transform formula for the is given as

$$F^{-1}[\bar{f}(\beta_m)] = f(x) = \sum_{m=1}^{\infty} \frac{K(\beta_m, x)}{N(\beta_m)} \bar{f}(\beta_m) \tag{33}$$

where $N(\beta_m)$ is the norm. The kernel $K(\beta_m, x)$ equals the eigenfunctions of the following equations based on the homogeneous analogue of the original partial differential equation (PDE) subject to the prescribed boundary conditions of the governing equation.

$$\frac{d^2 K(\beta_m, x)}{dx^2} + \beta_m^2 K(\beta_m, x) = 0 \tag{34}$$

For two-dimensional problem, the generalised finite Fourier transform of a real function *f(x,y)* and its inversion formulas are defined as

$$F[f(x, y)] = \bar{f}(\beta_m, \gamma_n) = \int_0^L \int_0^B K(\beta_m, x) K(\gamma_p, y) f(x, y)dxdy \tag{35}$$

The general inversion Finite Fourier transform formula is given as

$$F^{-1}[\bar{f}(\beta_m, \gamma_n)] = f(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{K(\beta_m, x) K(\beta_n, y)}{N(\beta_m) N(\gamma_n)} \bar{f}(\beta_m, \gamma_n) \tag{36}$$

The kernels $K(\beta_m, x)$ and $K(\gamma_p, y)$ equals the eigenfunctions of the following equations based on the homogeneous analogue of the original PDE subject to the prescribed boundary conditions of the governing equation.

$$\frac{d^2K(\beta_m,x)}{dx^2}+\beta_m^2K(\beta_m,x)=0 \tag{37}$$

$$\frac{d^2K(\gamma_n,x)}{dx^2}+\gamma_n^2K(\gamma_n,x)=0 \tag{38}$$

To apply the integral transform, the thermal model is expressed as:

#### 4.2.1. Layer 1: Epidermis Layer

$$\begin{aligned}&\rho_1c'_{p,1}\frac{\partial T_1}{\partial t}+\tau_{q,1}\rho_1c'_{p,1}\frac{\partial^2T_l}{\partial t^2}+\frac{\tau_{q,1}^2}{2}\rho_1c'_{p,1}\frac{\partial^3T_1}{\partial t^3}+\omega_bc_b\tau_{q,1}\frac{\partial T_1}{\partial t}+\omega_bc_b\frac{\tau_{q,1}^2}{2}\frac{\partial^2T_1}{\partial t^2}\\&=k_1\left(\left(\frac{\partial^2T_1}{\partial x^2}+\frac{\partial^2T_1}{\partial z^2}\right)+\tau_{T_1}\left(\frac{\partial^3T_1}{\partial t\partial x^2}+\frac{\partial^3T_1}{\partial t\partial z^2}\right)+\frac{\tau_{T_1}^2}{2}\left(\frac{\partial^4T_1}{\partial t^2\partial x^2}+\frac{\partial^4T_1}{\partial t^2\partial z^2}\right)\right)\\&+\omega_bc_b\left(T_b-T_1\right)+q_m+\tau_{q,1}\frac{\partial q_m}{\partial t}+\frac{\tau_{q,1}^2}{2}\frac{\partial^2q_m}{\partial t^2}+q_{ext}+\tau_{q,1}\frac{\partial q_{ext}}{\partial t}+\frac{\tau_{q,1}^2}{2}\frac{\partial^2q_{ext}}{\partial t^2}\end{aligned} \tag{39}$$

#### 4.2.2. Layer 2: Dermis Layer

$$\begin{aligned}&\rho_2c'_{p,2}\frac{\partial T_2}{\partial t}+\tau_{q,2}\rho_2c'_{p,2}\frac{\partial^2T_2}{\partial t^2}+\frac{\tau_{q,2}^2}{2}\rho_2c'_{p,2}\frac{\partial^3T_2}{\partial t^3}+\omega_bc_b\tau_{q,2}\frac{\partial T_2}{\partial t}+\omega_bc_b\frac{\tau_{q,2}^2}{2}\frac{\partial^2T_2}{\partial t^2}\\&=k_2\left(\left(\frac{\partial^2T_2}{\partial x^2}+\frac{\partial^2T_2}{\partial z^2}\right)+\tau_{T_2}\left(\frac{\partial^3T_2}{\partial t\partial x^2}+\frac{\partial^3T_2}{\partial t\partial z^2}\right)+\frac{\tau_{T_2}^2}{2}\left(\frac{\partial^4T_2}{\partial t^2\partial x^2}+\frac{\partial^4T_2}{\partial t^2\partial z^2}\right)\right)\\&+\omega_bc_b\left(T_b-T_2\right)+q_m+\tau_{q,2}\frac{\partial q_m}{\partial t}+\frac{\tau_{q,2}^2}{2}\frac{\partial^2q_m}{\partial t^2}+q_{ext}+\tau_{q,2}\frac{\partial q_{ext}}{\partial t}+\frac{\tau_{q,2}^2}{2}\frac{\partial^2q_{ext}}{\partial t^2}\end{aligned} \tag{40}$$

#### 4.2.3. Layer 3: Subcutaneous Layer

$$\begin{aligned}&\rho_3c'_{p,3}\frac{\partial T_3}{\partial t}+\tau_{q,3}\rho_3c'_{p,3}\frac{\partial^2T_3}{\partial t^2}+\frac{\tau_{q,3}^2}{2}\rho_3c'_{p,3}\frac{\partial^3T_3}{\partial t^3}+\omega_bc_b\tau_{q,3}\frac{\partial T_3}{\partial t}+\omega_bc_b\frac{\tau_{q,3}^2}{2}\frac{\partial^2T_3}{\partial t^2}\\&=k_3\left(\left(\frac{\partial^2T_3}{\partial x^2}+\frac{\partial^2T_3}{\partial z^2}\right)+\tau_{T3}\left(\frac{\partial^3T_3}{\partial t\partial x^2}+\frac{\partial^3T_3}{\partial t\partial z^2}\right)+\frac{\tau_{T_3}^2}{2}\left(\frac{\partial^4T_3}{\partial t^2\partial x^2}+\frac{\partial^4T_3}{\partial t^2\partial z^2}\right)\right)\\&+\omega_bc_b\left(T_b-T_3\right)+q_m+\tau_{q,3}\frac{\partial q_m}{\partial t}+\frac{\tau_{q,3}^2}{2}\frac{\partial^2q_m}{\partial t^2}+q_{ext}+\tau_{q,3}\frac{\partial q_{ext}}{\partial t}+\frac{\tau_{q,3}^2}{2}\frac{\partial^2q_{ext}}{\partial t^2}\end{aligned} \tag{41}$$

Applying Laplace Transform to Eqs. (39) – (41)

$$
\begin{aligned}
&\rho_1 c'_{p,1}\left(s\overline{T_1}-T_b\right)+\tau_{q,1}\rho_1 c'_{p,1}\left(s^2\overline{T_1}-sT_b\right)+\frac{\tau_{q,1}^2}{2}\rho_1 c'_{p,1}\left(s^3\overline{T_1}-s^2T_b\right)\\
&+\omega_b c_b\tau_{q,1}\left(s\overline{T_1}-T_b\right)+\omega_b c_b\frac{\tau_{q,1}^2}{2}\left(s^2\overline{T_1}-sT_b\right)\\
&=k_1\left(\begin{aligned}&\left(\frac{\partial^2\overline{T}_1}{\partial x^2}+\frac{\partial^2\overline{T}_1}{\partial z^2}\right)+\tau_{T_1}\left[s\left(\frac{\partial^2\overline{T}_1}{\partial x^2}+\frac{\partial^2\overline{T}_1}{\partial z^2}\right)-\left(\frac{\partial^2\overline{T}_1}{\partial x^2}+\frac{\partial^2 T_1}{\partial z^2}\right)_{t=0}\right]\\
&+\frac{\tau_{T_1}^2}{2}\left[s^2\left(\frac{\partial^2\overline{T}_1}{\partial x^2}+\frac{\partial^2\overline{T}_1}{\partial z^2}\right)-s\left(\frac{\partial^2 T_1}{\partial x^2}+\frac{\partial^2 T_1}{\partial z^2}\right)_{t=0}-\left(\frac{\partial^2 T_1}{\partial x^2}+\frac{\partial^2 T_1}{\partial z^2}\right)_{t=0}\right]\end{aligned}\right)\\
&+\omega_b c_b\left(T_b-\overline{T}_1\right)+\overline{q}_m+\tau_{q,1}\left(s\overline{q}_m-q_{b,0}\right)+\frac{\tau_{q,1}^2}{2}\left(s^2\overline{q}_m-sq_{b,0}-\dot{q}_{b,0}\right)\\
&+\overline{q}_{ext}+\tau_{q,1}\left(s\overline{q}_{ext}-q_{ext,0}\right)+\frac{\tau_{q,1}^2}{2}\left(s^2\overline{q}_{ext}-sq_{ext,0}-\dot{q}_{ext,0}\right)
\end{aligned}
\tag{42}
$$

$$
\begin{aligned}
&\rho_2 c'_{p,2}\left(s\overline{T_2}-T_b\right)+\tau_{q,2}\rho_2 c'_{p,2}\left(s^2\overline{T_2}-sT_b\right)+\frac{\tau_{q,2}^2}{2}\rho_2 c'_{p,2}\left(s^3\overline{T_2}-s^2T_b\right)\\
&+\omega_b c_b\tau_{q,2}\left(s\overline{T_2}-T_b\right)+\omega_b c_b\frac{\tau_{q,2}^2}{2}\left(s^2\overline{T_2}-sT_b\right)\\
&=k_2\left(\begin{aligned}&\left(\frac{\partial^2\overline{T}_2}{\partial x^2}+\frac{\partial^2\overline{T}_2}{\partial z^2}\right)+\tau_{T_2}\left[s\left(\frac{\partial^2\overline{T}_2}{\partial x^2}+\frac{\partial^2\overline{T}_2}{\partial z^2}\right)-\left(\frac{\partial^2\overline{T}_2}{\partial x^2}+\frac{\partial^2 T_2}{\partial z^2}\right)_{t=0}\right]\\
&+\frac{\tau_{T_2}^2}{2}\left[s^2\left(\frac{\partial^2\overline{T}_2}{\partial x^2}+\frac{\partial^2\overline{T}_2}{\partial z^2}\right)-s\left(\frac{\partial^2 T_2}{\partial x^2}+\frac{\partial^2 T_2}{\partial z^2}\right)_{t=0}-\left(\frac{\partial^2 T_2}{\partial x^2}+\frac{\partial^2 T_2}{\partial z^2}\right)_{t=0}\right]\end{aligned}\right)\\
&+\omega_b c_b\left(T_b-\overline{T}_2\right)+\overline{q}_m+\tau_{q,2}\left(s\overline{q}_m-q_{b,0}\right)+\frac{\tau_{q,2}^2}{2}\left(s^2\overline{q}_m-sq_{b,0}-\dot{q}_{b,0}\right)\\
&+\overline{q}_{ext}+\tau_{q,2}\left(s\overline{q}_{ext}-q_{ext,0}\right)+\frac{\tau_{q,2}^2}{2}\left(s^2\overline{q}_{ext}-sq_{ext,0}-\dot{q}_{ext,0}\right)
\end{aligned}
\tag{43}
$$

$$
\begin{aligned}
&\rho_3 c'_{p,3}\left(s\overline{T_3}-T_b\right)+\tau_{q,3}\rho_3 c'_{p,3}\left(s^2\overline{T_3}-sT_b\right)+\frac{\tau_{q,3}^2}{2}\rho_3 c'_{p,3}\left(s^3\overline{T_3}-s^2T_b\right)\\
&+\omega_b c_b\tau_{q,3}\left(s\overline{T_3}-T_b\right)+\omega_b c_b\frac{\tau_{q,3}^2}{2}\left(s^2\overline{T_3}-sT_b\right)\\
&=k_3\left(\begin{aligned}&\left(\frac{\partial^2\overline{T}_3}{\partial x^2}+\frac{\partial^2\overline{T}_3}{\partial z^2}\right)+\tau_{T_3}\left[s\left(\frac{\partial^2\overline{T}_3}{\partial x^2}+\frac{\partial^2\overline{T}_3}{\partial z^2}\right)-\left(\frac{\partial^2\overline{T}_3}{\partial x^2}+\frac{\partial^2 T_3}{\partial z^2}\right)_{t=0}\right]\\
&+\frac{\tau_{T_3}^2}{2}\left[s^2\left(\frac{\partial^2\overline{T}_3}{\partial x^2}+\frac{\partial^2\overline{T}_3}{\partial z^2}\right)-s\left(\frac{\partial^2 T_3}{\partial x^2}+\frac{\partial^2 T_3}{\partial z^2}\right)_{t=0}-\left(\frac{\partial^2 T_3}{\partial x^2}+\frac{\partial^2 T_3}{\partial z^2}\right)_{t=0}\right]\end{aligned}\right)\\
&+\omega_b c_b\left(T_b-\overline{T}_3\right)+\overline{q}_m+\tau_{q,3}\left(s\overline{q}_m-q_{b,0}\right)+\frac{\tau_{q,3}^2}{2}\left(s^2\overline{q}_m-sq_{b,0}-\dot{q}_{b,0}\right)\\
&+\overline{q}_{ext}+\tau_{q,3}\left(s\overline{q}_m-q_{ext,0}\right)+\frac{\tau_{q,3}^2}{2}\left(s^2\overline{q}_{ext}-sq_{ext,0}-\dot{q}_{ext,0}\right)
\end{aligned}
\tag{44}
$$

Collecting like terms in Eqs. (42) – (44) gives

$$
\begin{aligned}
&k_1\left(\frac{\tau_{T_1}^2}{2}s^2+\tau_{T_1}s+1\right)\left(\frac{\partial^2\overline{T}_1}{\partial x^2}+\frac{\partial^2\overline{T}_1}{\partial z^2}\right)\\
&-\left(\frac{\tau_{q,1}^2}{2}\rho_1 c'_{p,1}s^3+\left(\tau_{q,1}\rho_1 c'_{p,1}+\omega_b c_b\frac{\tau_{q,1}^2}{2}\right)s^2+\left(\rho_1 c'_{p,1}+\omega_b c_b\tau_{q,1}\right)s+\omega_b c_b\right)\overline{T}_1\\
&=-\left(\frac{\tau_{q,1}^2}{2}\rho_1 c'_{p,1}s^2+\tau_{q,1}\rho_1 c'_{p,1}s+\rho_1 c'_{p,1}\right)T_b-\left(\omega_b c_b\tau_{q,1}s+\omega_b c_b\frac{\tau_{q,1}^2}{2}\right)T_b-\omega_b c_b T_b\\
&-k_1\left\{\left[\tau_{T_1}\left(\frac{\partial^2 T_1}{\partial x^2}+\frac{\partial^2 T_1}{\partial z^2}\right)\bigg|_{t=0}\right]+s\frac{\tau_{T_1}^2}{2}\left(\frac{\partial^2 T_1}{\partial x^2}+\frac{\partial^2 T_1}{\partial z^2}\right)\bigg|_{t=0}+\frac{\tau_{T_1}^2}{2}\left(\frac{\partial^2 T_1}{\partial x^2}+\frac{\partial^2 T_1}{\partial z^2}\right)\bigg|_{t=0}\right\}\\
&-\left[\begin{array}{l}\overline{q}_m+\tau_{q,1}\left(s\overline{q}_m-q_{b,0}\right)+\dfrac{\tau_{q,1}^2}{2}\left(s^2\overline{q}_m-sq_{b,0}-\dot{q}_{b,0}\right)\\ +\overline{q}_{ext}+\tau_{q,1}\left(s\overline{q}_{ext}-q_{\mathrm{ext},0}\right)+\dfrac{\tau_{q,1}^2}{2}\left(s^2\overline{q}_{ext}-sq_{\mathrm{ext},0}-\dot{q}_{\mathrm{ext},0}\right)\end{array}\right]
\end{aligned}
\tag{45}
$$

$$
\begin{aligned}
&k_2\left(\frac{\tau_{T_2}^2}{2}s^2+\tau_{T_2}s+1\right)\left(\frac{\partial^2\overline{T}_2}{\partial x^2}+\frac{\partial^2\overline{T}_2}{\partial z^2}\right)\\
&-\left(\frac{\tau_{q,2}^2}{2}\rho_2 c'_{p,2}s^3+\left(\tau_{q,2}\rho_2 c'_{p,2}+\omega_b c_b\frac{\tau_{q,2}^2}{2}\right)s^2+\left(\rho_2 c'_{p,2}+\omega_b c_b\tau_{q,2}\right)s+\omega_b c_b\right)\overline{T}_2\\
&=-\left(\frac{\tau_{q,2}^2}{2}\rho_2 c'_{p,2}s^2+\tau_{q,2}\rho_2 c'_{p,2}s+\rho_2 c'_{p,2}\right)T_b-\left(\omega_b c_b\tau_{q,2}s+\omega_b c_b\frac{\tau_{q,2}^2}{2}\right)T_b-\omega_b c_b T_b\\
&-k_1\left\{\left[\tau_{T_2}\left(\frac{\partial^2 T_2}{\partial x^2}+\frac{\partial^2 T_2}{\partial z^2}\right)\bigg|_{t=0}\right]+s\frac{\tau_{T_2}^2}{2}\left(\frac{\partial^2 T_2}{\partial x^2}+\frac{\partial^2 T_2}{\partial z^2}\right)\bigg|_{t=0}+\frac{\tau_{T_2}^2}{2}\left(\frac{\partial^2 T_2}{\partial x^2}+\frac{\partial^2 T_2}{\partial z^2}\right)\bigg|_{t=0}\right\}\\
&-\left[\begin{array}{l}\overline{q}_m+\tau_{q,2}\left(s\overline{q}_m-q_{b,0}\right)+\dfrac{\tau_{q,2}^2}{2}\left(s^2\overline{q}_m-sq_{b,0}-\dot{q}_{b,0}\right)\\ +\overline{q}_{ext}+\tau_{q,2}\left(s\overline{q}_{ext}-q_{\mathrm{ext},0}\right)+\dfrac{\tau_{q,2}^2}{2}\left(s^2\overline{q}_{ext}-sq_{\mathrm{ext},0}-\dot{q}_{\mathrm{ext},0}\right)\end{array}\right]
\end{aligned}
\tag{46}
$$

$$
\begin{aligned}
&k_3\left(\frac{\tau_{T_3}^2}{2}s^2+\tau_{T_3}s+1\right)\left(\frac{\partial^2\overline{T}_3}{\partial x^2}+\frac{\partial^2\overline{T}_3}{\partial z^2}\right)\\
&-\left(\frac{\tau_{q,3}^2}{2}\rho_3 c'_{p,3}s^3+\left(\tau_{q,3}\rho_3 c'_{p,3}+\omega_b c_b\frac{\tau_{q,3}^2}{2}\right)s^2+\left(\rho_3 c'_{p,3}+\omega_b c_b\tau_{q,3}\right)s+\omega_b c_b\right)\overline{T}_3\\
&=-\left(\frac{\tau_{q,3}^2}{2}\rho_3 c'_{p,3}s^2+\tau_{q,3}\rho_3 c'_{p,3}s+\rho_3 c'_{p,3}\right)T_b-\left(\omega_b c_b\tau_{q,3}s+\omega_b c_b\frac{\tau_{q,3}^2}{2}\right)T_b-\omega_b c_b T_b\\
&-k_1\left\{\left[\tau_{T_3}\left(\frac{\partial^2 T_3}{\partial x^2}+\frac{\partial^2 T_3}{\partial z^2}\right)\bigg|_{t=0}\right]+s\frac{\tau_{T_3}^2}{2}\left(\frac{\partial^2 T_3}{\partial x^2}+\frac{\partial^2 T_3}{\partial z^2}\right)\bigg|_{t=0}+\frac{\tau_{T_3}^2}{2}\left(\frac{\partial^2 T_3}{\partial x^2}+\frac{\partial^2 T_3}{\partial z^2}\right)\bigg|_{t=0}\right\}\\
&-\left[\begin{array}{l}\overline{q}_m+\tau_{q,3}\left(s\overline{q}_m-q_{b,0}\right)+\dfrac{\tau_{q,3}^2}{2}\left(s^2\overline{q}_m-sq_{b,0}-\dot{q}_{b,0}\right)\\ +\overline{q}_{ext}+\tau_{q,1}\left(s\overline{q}_{ext}-q_{\mathrm{ext},0}\right)+\dfrac{\tau_{q,3}^2}{2}\left(s^2\overline{q}_{ext}-sq_{\mathrm{ext},0}-\dot{q}_{\mathrm{ext},0}\right)\end{array}\right]
\end{aligned}
\tag{47}
$$

It should be noted that initially, there is no temperature gradient and heat gradient at any point in the skin tissue. Therefore, the above equation reduces to

$$
\begin{aligned}
&k_1\left(\frac{\tau_{T_1}^2}{2}s^2+\tau_{T_1}s+1\right)\left(\frac{\partial^2\overline{T}_1}{\partial x^2}+\frac{\partial^2\overline{T}_1}{\partial z^2}\right)\\
&-\left(\frac{\tau_{q,1}^2}{2}\rho_1c'_{p,1}s^3+\left(\tau_{q,1}\rho_1c'_{p,1}+\omega_bc_b\frac{\tau_{q,1}^2}{2}\right)s^2+\left(\rho_1c'_{p,1}+\omega_bc_b\tau_{q,1}\right)s+\omega_bc_b\right)\overline{T}_1\\
&=-\left(\frac{\tau_{q,1}^2}{2}\rho_1c'_{p,1}s^2+\tau_{q,1}\rho_1c'_{p,1}s+\rho_1c_{p,1}\right)T_b-\left(\omega_bc_b\tau_{q,1}s+\omega_bc_b\frac{\tau_{q,1}^2}{2}\right)T_b-\omega_bc_bT_b\\
&-\left[\overline{q}_m+\tau_{q,1}\left(s\overline{q}_m-q_{b,0}\right)+\frac{\tau_{q,1}^2}{2}\left(s^2\overline{q}_m-sq_{b,0}\right)+\overline{q}_{ext}+\tau_{q,1}\left(s\overline{q}_{ext}-q_{\text{ext},0}\right)+\frac{\tau_{q,1}^2}{2}\left(s^2\overline{q}_{ext}-sq_{\text{ext},0}\right)\right]
\end{aligned}
\tag{48}
$$

$$
\begin{aligned}
&k_2\left(\frac{\tau_{T_2}^2}{2}s^2+\tau_{T_2}s+1\right)\left(\frac{\partial^2\overline{T}_2}{\partial x^2}+\frac{\partial^2\overline{T}_2}{\partial z^2}\right)\\
&-\left(\frac{\tau_{q,2}^2}{2}\rho_2c'_{p,2}s^3+\left(\tau_{q,2}\rho_2c'_{p,2}+\omega_bc_b\frac{\tau_{q,2}^2}{2}\right)s^2+\left(\rho_2c'_{p,2}+\omega_bc_b\tau_{q,2}\right)s+\omega_bc_b\right)\overline{T}_2\\
&=-\left(\frac{\tau_{q,2}^2}{2}\rho_2c'_{p,2}s^2+\tau_{q,2}\rho_2c'_{p,2}s+\rho_2c'_{p,2}\right)T_b-\left(\omega_bc_b\tau_{q,2}s+\omega_bc_b\frac{\tau_{q,2}^2}{2}\right)T_b-\omega_bc_bT_b\\
&-\left[\overline{q}_m+\tau_{q,2}\left(s\overline{q}_m-q_{b,0}\right)+\frac{\tau_{q,2}^2}{2}\left(s^2\overline{q}_m-sq_{b,0}\right)+\overline{q}_{ext}+\tau_{q,2}\left(s\overline{q}_{ext}-q_{\text{ext},0}\right)+\frac{\tau_{q,2}^2}{2}\left(s^2\overline{q}_{ext}-sq_{\text{ext},0}\right)\right]
\end{aligned}
\tag{49}
$$

$$
\begin{aligned}
&k_3\left(\frac{\tau_{T_3}^2}{2}s^2+\tau_{T_3}s+1\right)\left(\frac{\partial^2\overline{T}_3}{\partial x^2}+\frac{\partial^2\overline{T}_3}{\partial z^2}\right)\\
&-\left(\frac{\tau_{q,3}^2}{2}\rho_3c'_{p,3}s^3+\left(\tau_{q,3}\rho_3c'_{p,3}+\omega_bc_b\frac{\tau_{q,3}^2}{2}\right)s^2+\left(\rho_3c'_{p,3}+\omega_bc_b\tau_{q,3}\right)s+\omega_bc_b\right)\overline{T}_3\\
&=-\left(\frac{\tau_{q,3}^2}{2}\rho_3c'_{p,3}s^2+\tau_{q,3}\rho_3c'_{p,3}s+\rho_3c'_{p,3}\right)T_b-\left(\omega_bc_b\tau_{q,3}s+\omega_bc_b\frac{\tau_{q,3}^2}{2}\right)T_b-\omega_bc_bT_b\\
&-\left[\overline{q}_m+\tau_{q,3}\left(s\overline{q}_m-q_{b,0}\right)+\frac{\tau_{q,3}^2}{2}\left(s^2\overline{q}_m-sq_{b,0}\right)+\overline{q}_{ext}+\tau_{q,1}\left(s\overline{q}_{ext}-q_{\text{ext},0}\right)+\frac{\tau_{q,3}^2}{2}\left(s^2\overline{q}_{ext}-sq_{\text{ext},0}\right)\right]
\end{aligned}
\tag{50}
$$

rearranging Eqs. (48) – (50) gives

$$
\begin{aligned}
&\left(\frac{\partial^2\overline{T}_1}{\partial x^2}+\frac{\partial^2\overline{T}_1}{\partial z^2}\right)-\frac{1}{k_1\left(\frac{\tau_{T_1}^2}{2}s^2+\tau_{T_1}s+1\right)}\begin{pmatrix}\frac{\tau_{q,1}^2}{2}\rho_1c'_{p,1}s^3+\left(\tau_{q,1}\rho_1c'_{p,1}+\omega_bc_b\frac{\tau_{q,1}^2}{2}\right)s^2\\+\left(\rho_1c'_{p,1}+\omega_bc_b\tau_{q,1}\right)s+\omega_bc_b\end{pmatrix}\overline{T}_1\\
&=-\frac{1}{k_1\left(\frac{\tau_{T_1}^2}{2}s^2+\tau_{T_1}s+1\right)}\begin{bmatrix}\left(\frac{\tau_{q,1}^2}{2}\rho_1c'_{p,1}s^2+\tau_{q,1}\rho_1c'_{p,1}s+\rho_1c_{p,1}\right)\\+\left(\omega_bc_b\tau_{q,1}s+\omega_bc_b\tau_{q,1}\frac{\tau_{q,1}^2}{2}\right)+\omega_bc_b\end{bmatrix}T_b\\
&-\frac{1}{k_1\left(\frac{\tau_{T_1}^2}{2}s^2+\tau_{T_1}s+1\right)}\begin{bmatrix}\overline{q}_m+\tau_{q,1}\left(s\overline{q}_m-q_{b,0}\right)+\frac{\tau_{q,1}^2}{2}\left(s^2\overline{q}_m-sq_{b,0}\right)\\+\overline{q}_{ext}+\tau_{q,1}\left(s\overline{q}_{ext}-q_{\text{ext},0}\right)+\frac{\tau_{q,1}^2}{2}\left(s^2\overline{q}_{ext}-sq_{\text{ext},0}\right)\end{bmatrix}
\end{aligned}
\tag{51}
$$

$$\left(\frac{\partial^2\overline{T}_2}{\partial x^2}+\frac{\partial^2\overline{T}_2}{\partial z^2}\right)-\frac{1}{k_2\left(\frac{\tau_{T_2}^2}{2}s^2+\tau_{T_2}s+1\right)}\left(\begin{array}{l}\frac{\tau_{q,2}^2}{2}\rho_2c'_{p,2}s^3+\left(\tau_{q,2}\rho_2c'_{p,2}+\omega_bc_b\frac{\tau_{q,2}^2}{2}\right)s^2\\+\left(\rho_2c'_{p,2}+\omega_bc_b\tau_{q,2}\right)s+\omega_bc_b\end{array}\right)\overline{T}_2$$

$$=-\frac{1}{k_2\left(\frac{\tau_{T_2}^2}{2}s^2+\tau_{T_2}s+1\right)}\left[\begin{array}{l}\left(\frac{\tau_{q,2}^2}{2}\rho_2c'_{p,2}s^2+\tau_{q,2}\rho_2c'_{p,2}s+\rho_2c'_{p,2}\right)\\+\left(\omega_bc_b\tau_{q,2}s+\omega_bc_b\frac{\tau_{q,2}^2}{2}\right)+\omega_bc_b\end{array}\right]T_b \tag{52}$$

$$-\frac{1}{k_2\left(\frac{\tau_{T_2}^2}{2}s^2+\tau_{T_2}s+1\right)}\left[\begin{array}{l}\overline{q}_m+\tau_{q,2}\left(s\overline{q}_m-q_{b,0}\right)+\frac{\tau_{q,2}^2}{2}\left(s^2\overline{q}_m-sq_{b,0}\right)\\+\overline{q}_{ext}+\tau_{q,2}\left(s\overline{q}_{ext}-q_{\mathrm{ext},0}\right)+\frac{\tau_{q,2}^2}{2}\left(s^2\overline{q}_{ext}-sq_{\mathrm{ext},0}\right)\end{array}\right]$$

$$\left(\frac{\partial^2\overline{T}_3}{\partial x^2}+\frac{\partial^2\overline{T}_3}{\partial z^2}\right)-\frac{1}{k_3\left(\frac{\tau_{T_3}^2}{2}s^2+\tau_{T_3}s+1\right)}\left(\begin{array}{l}\frac{\tau_{q,3}^2}{2}\rho_3c'_{p,3}s^3+\left(\tau_{q,3}\rho_3c'_{p,3}+\omega_bc_b\frac{\tau_{q,3}^2}{2}\right)s^2\\+\left(\rho_3c'_{p,3}+\omega_bc_b\tau_{q,3}\right)s+\omega_bc_b\end{array}\right)\overline{T}_3$$

$$=-\frac{1}{k_3\left(\frac{\tau_{T_3}^2}{2}s^2+\tau_{T_3}s+1\right)}\left[\begin{array}{l}\left(\frac{\tau_{q,3}^2}{2}\rho_3c'_{p,3}s^2+\tau_{q,3}\rho_3c'_{p,3}s+\rho_3c_{p,3}\right)\\+\left(\omega_bc_b\tau_{q,3}s+\omega_bc_b\frac{\tau_{q,3}^2}{2}\right)+\omega_bc_b\end{array}\right]T_b \tag{53}$$

$$-\frac{1}{k_3\left(\frac{\tau_{T_3}^2}{2}s^2+\tau_{T_3}s+1\right)}\left[\begin{array}{l}\overline{q}_m+\tau_{q,3}\left(s\overline{q}_m-q_{b,0}\right)+\frac{\tau_{q,3}^2}{2}\left(s^2\overline{q}_m-sq_{b,0}\right)\\+\overline{q}_{ext}+\tau_{q,1}\left(s\overline{q}_{ext}-q_{\mathrm{ext},0}\right)+\frac{\tau_{q,3}^2}{2}\left(s^2\overline{q}_{ext}-sq_{\mathrm{ext},0}\right)\end{array}\right]$$

Further rearranging Eqs. (51) – (52) for the simplification of Laplace inversion produces

$$\left(\frac{\partial^2\overline{T}_1}{\partial x^2}+\frac{\partial^2\overline{T}_1}{\partial z^2}\right)-\frac{\left(\tau_{q,1}^2\rho_1c'_{p,1}\right)}{k_1\tau_{T_1}^2\left(s^2+\frac{2}{\tau_{T_1}}s+\frac{2}{\tau_{T_1}^2}\right)}\left(\begin{array}{l}s^3+\frac{2\left(\tau_{q,1}\rho_1c'_{p,1}+\omega_bc_b\frac{\tau_{q,1}^2}{2}\right)}{\tau_{q,1}^2\rho_1c'_{p,1}}s^2\\+\frac{2\left(\rho_1c'_{p,1}+\omega_bc_b\tau_{q,1}\right)}{\tau_{q,1}^2\rho_1c'_{p,1}}s+\frac{2\omega_bc_b}{\tau_{q,1}^2\rho_1c'_{p,1}}\end{array}\right)\overline{T}_1$$

$$=-\frac{\tau_{q,1}^2\rho_1c'_{p,1}}{k_1\tau_{T_1}^2\left(s^2+\frac{2}{\tau_{T_1}}s+\frac{2}{\tau_{T_1}^2}\right)}\left[\left(s^2+\frac{2}{\tau_{q,1}}s+\frac{2}{\tau_{q,1}^2}\right)+\frac{2\omega_bc_b\left(s+\frac{\tau_{q,1}}{2}\right)}{\tau_{q,1}\rho_1c'_{p,1}}+\frac{2\omega_bc_b}{\tau_{q,1}^2\rho_1c'_{p,1}}\right]T_b \tag{54}$$

$$-\frac{\tau_{q,1}^2}{k_1\tau_{T_1}^2\left(s^2+\frac{2}{\tau_{T_1}}s+\frac{2}{\tau_{T_1}^2}\right)}\left[\begin{array}{l}\left(s^2+\frac{2}{\tau_{q,1}}s+\frac{2}{\tau_{q,1}^2}\right)\overline{q}_m-\left(s+\frac{2}{\tau_{q,1}}\right)q_{b,0}\\\left(s^2+\frac{2}{\tau_{q,1}}s+\frac{2}{\tau_{q,1}^2}\right)\overline{q}_{ext}-\left(s+\frac{2}{\tau_{q,1}}\right)q_{\mathrm{ext},0}\end{array}\right]$$

$$\left(\frac{\partial^2\overline{T}_2}{\partial x^2}+\frac{\partial^2\overline{T}_2}{\partial z^2}\right)-\frac{\left(\tau_{q,2}^2\rho_2 c'_{p,2}\right)}{k_2\tau_{T_2}^2\left(s^2+\frac{2}{\tau_{T_2}}s+\frac{2}{\tau_{T_2}^2}\right)}\left(\begin{array}{l}s^3+\frac{2\left(\tau_{q,2}\rho_2 c'_{p,2}+\omega_b c_b\frac{\tau_{q,2}^2}{2}\right)}{\tau_{q,2}^2\rho_2 c'_{p,2}}s^2\\+\frac{2\left(\rho_2 c'_{p,2}+\omega_b c_b\tau_{q,2}\right)}{\tau_{q,2}^2\rho_2 c'_{p,2}}s+\frac{2\omega_b c_b}{\tau_{q,2}^2\rho_2 c'_{p,2}}\end{array}\right)\overline{T}_2$$

$$=-\frac{\tau_{q,2}^2\rho_2 c'_{p,2}}{k_2\tau_{T_2}^2\left(s^2+\frac{2}{\tau_{T_2}}s+\frac{2}{\tau_{T_2}^2}\right)}\left[\left(s^2+\frac{2}{\tau_{q,2}}s+\frac{2}{\tau_{q,2}^2}\right)+\frac{2\omega_b c_b\left(s+\frac{\tau_{q,2}}{2}\right)}{\tau_{q,2}\rho_2 c'_{p,2}}+\frac{2\omega_b c_b}{\tau_{q,2}^2\rho_2 c'_{p,2}}\right]T_b \tag{55}$$

$$-\frac{\tau_{q,2}^2}{k_2\tau_{T_2}^2\left(s^2+\frac{2}{\tau_{T_2}}s+\frac{2}{\tau_{T_2}^2}\right)}\left[\begin{array}{l}\left(s^2+\frac{2}{\tau_{q,2}}s+\frac{2}{\tau_{q,2}^2}\right)\overline{q}_m-\left(s+\frac{2}{\tau_{q,2}}\right)q_{b,0}\\\left(s^2+\frac{2}{\tau_{q,2}}s+\frac{2}{\tau_{q,2}^2}\right)\overline{q}_{ext}-\left(s+\frac{2}{\tau_{q,2}}\right)q_{\mathrm{ext},0}\end{array}\right]$$

$$\left(\frac{\partial^2\overline{T}_3}{\partial x^2}+\frac{\partial^2\overline{T}_3}{\partial z^2}\right)-\frac{\left(\tau_{q,3}^2\rho_3 c'_{p,3}\right)}{k_3\tau_{T_3}^2\left(s^2+\frac{2}{\tau_{T_3}}s+\frac{2}{\tau_{T_3}^2}\right)}\left(\begin{array}{l}s^3+\frac{2\left(\tau_{q,3}\rho_3 c'_{p,3}+\omega_b c_b\frac{\tau_{q,3}^2}{2}\right)}{\tau_{q,3}^2\rho_3 c'_{p,3}}s^2\\+\frac{2\left(\rho_3 c'_{p,3}+\omega_b c_b\tau_{q,3}\right)}{\tau_{q,3}^2\rho_3 c'_{p,3}}s+\frac{2\omega_b c_b}{\tau_{q,3}^2\rho_3 c'_{p,3}}\end{array}\right)\overline{T}_3$$

$$=-\frac{\tau_{q,3}^2\rho_3 c'_{p,3}}{k_3\tau_{T_3}^2\left(s^2+\frac{2}{\tau_{T_3}}s+\frac{2}{\tau_{T_3}^2}\right)}\left[\left(s^2+\frac{2}{\tau_{q,3}}s+\frac{2}{\tau_{q,3}^2}\right)+\frac{2\omega_b c_b\left(s+\frac{\tau_{q,3}}{2}\right)}{\tau_{q,3}\rho_3 c'_{p,3}}+\frac{2\omega_b c_b}{\tau_{q,3}^2\rho_3 c'_{p,3}}\right]T_b \tag{56}$$

$$-\frac{\tau_{q,3}^2}{k_3\tau_{T_3}^2\left(s^2+\frac{2}{\tau_{T_3}}s+\frac{2}{\tau_{T_3}^2}\right)}\left[\begin{array}{l}\left(s^2+\frac{2}{\tau_{q,3}}s+\frac{2}{\tau_{q,3}^2}\right)\overline{q}_m-\left(s+\frac{2}{\tau_{q,3}}\right)q_{b,0}\\\left(s^2+\frac{2}{\tau_{q,3}}s+\frac{2}{\tau_{q,3}^2}\right)\overline{q}_{ext}-\left(s+\frac{2}{\tau_{q,3}}\right)q_{\mathrm{ext},0}\end{array}\right]$$

**4.3.1. The boundary conditions in the Laplace domain are:**

$$s>0,\ x=0,\ 0\le z\le H,\ \frac{\partial\overline{T}_1}{\partial x}=0 \tag{57a}$$

$$s>0,\ x=0,\ 0\le z\le H,\ \frac{\partial\overline{T}_2}{\partial x}=0 \tag{57b}$$

$$s>0,\ x=0,\ 0\le z\le H,\ \frac{\partial\overline{T}_3}{\partial x}=0 \tag{57c}$$

$$s>0,\ x=L,\ 0\le z\le H,\ \frac{\partial\overline{T}_1}{\partial x}=0 \tag{57d}$$

$$s>0,\ x=L,\ 0\le z\le H,\ \frac{\partial\overline{T}_2}{\partial x}=0 \tag{57e}$$

$$s > 0,\ x = L,\ 0 \le z \le H,\ \frac{\partial \overline{T_3}}{\partial x} = 0 \tag{57f}$$

$$s > 0,\ z = 0,\ 0 \le x \le L - k\frac{\partial \overline{T_1}}{\partial z} = h\left(\frac{T_a}{s} - \overline{T_1}\right) \tag{57g}$$

$$s > 0,\ z = H,\ 0 \le x \le L,\ \overline{T_3} = \frac{T_b}{s} \tag{57h}$$

**4.3.2. Interlayer conditions in Laplace domain are:**

$$s > 0,\ z = z_1,\ 0 \le x \le L,\ \overline{T_1} = \overline{T_2}, \tag{58a}$$

$$s > 0,\ z = z_1,\ 0 \le x \le L,\ k_1\frac{\partial \overline{T_1}}{\partial z} = k_2\frac{\partial \overline{T_2}}{\partial z} \tag{58b}$$

$$s > 0,\ z = z_2,\ 0 \le x \le L,\ \overline{T_2} = \overline{T_3}, \tag{58c}$$

$$s > 0,\ z = z_1,\ 0 \le x \le L,\ k_2\frac{\partial \overline{T_2}}{\partial z} = k_3\frac{\partial \overline{T_3}}{\partial z}, \tag{58d}$$

Further rearranging Eqs. (54) – (56) for the simplification of Laplace inversion produces

$$\begin{aligned}
&\left(-\beta_{m1}^2\tilde{\tilde{T}}_1 + \frac{d^2\tilde{\tilde{T}}_1}{dz^2}\right) - \frac{\left(\tau_{q,1}^2\rho_1 c'_{p,1}\right)}{k_1\tau_{T_1}^2\left(s^2 + \frac{2}{\tau_{T_1}}s + \frac{2}{\tau_{T_1}^2}\right)}\left(\begin{aligned}&s^3 + \frac{2\left(\tau_{q,1}\rho_1 c'_{p,1} + \omega_b c_b\frac{\tau_{q,1}^2}{2}\right)}{\tau_{q,1}^2\rho_1 c'_{p,1}}s^2\\ &+\frac{2\left(\rho_1 c'_{p,1} + \omega_b c_b\tau_{q,1}\right)}{\tau_{q,1}^2\rho_1 c'_{p,1}}s + \frac{2\omega_b c_b}{\tau_{q,1}^2\rho_1 c'_{p,1}}\end{aligned}\right)\tilde{\tilde{T}}_1\\
&= -\frac{\tau_{q,1}^2\rho_1 c'_{p,1}}{k_1\tau_{T_1}^2\left(s^2 + \frac{2}{\tau_{T_1}}s + \frac{2}{\tau_{T_1}^2}\right)}\left[\left(s^2 + \frac{2}{\tau_{q,1}}s + \frac{2}{\tau_{q,1}^2}\right) + \frac{2\omega_b c_b\left(s + \frac{\tau_{q,1}}{2}\right)}{\tau_{q,1}\rho_1 c'_{p,1}} + \frac{2\omega_b c_b}{\tau_{q,1}^2\rho_1 c'_{p,1}}\right]\tilde{T}_b\\
&-\frac{\tau_{q,1}^2}{k_1\tau_{T_1}^2\left(s^2 + \frac{2}{\tau_{T_1}}s + \frac{2}{\tau_{T_1}^2}\right)}\left[\begin{aligned}&\left(s^2 + \frac{2}{\tau_{q,1}}s + \frac{2}{\tau_{q,1}^2}\right)\tilde{\tilde{q}}_m - \left(s + \frac{2}{\tau_{q,1}}\right)\tilde{q}_{b,0}\\ &\left(s^2 + \frac{2}{\tau_{q,1}}s + \frac{2}{\tau_{q,1}^2}\right)\tilde{\tilde{q}}_{ext} - \left(s + \frac{2}{\tau_{q,1}}\right)\tilde{q}_{\text{ext},0}\end{aligned}\right]
\end{aligned} \tag{59}$$

$$
\left(-\beta_{m2}^{2}\tilde{\tilde{T}}_{2}+\frac{d^{2}\tilde{\tilde{T}}_{2}}{dz^{2}}\right)-\frac{\left(\tau_{q,2}^{2}\rho_{2}c'_{p,2}\right)}{k_{2}\tau_{T_{2}}^{2}\left(s^{2}+\frac{2}{\tau_{T_{2}}}s+\frac{2}{\tau_{T_{2}}^{2}}\right)}\left(\begin{array}{l}s^{3}+\frac{2\left(\tau_{q,2}\rho_{2}c'_{p,2}+\omega_{b}c_{b}\frac{\tau_{q,2}^{2}}{2}\right)}{\tau_{q,2}^{2}\rho_{2}c'_{p,2}}s^{2}\\+\frac{2\left(\rho_{2}c'_{p,2}+\omega_{b}c_{b}\tau_{q,2}\right)}{\tau_{q,2}^{2}\rho_{2}c'_{p,2}}s+\frac{2\omega_{b}c_{b}}{\tau_{q,2}^{2}\rho_{2}c'_{p,2}}-\beta_{m2}^{2}\tilde{\tilde{T}}_{2}\end{array}\right)\tilde{\tilde{T}}_{2}
$$

$$
=-\frac{\tau_{q,2}^{2}\rho_{2}c'_{p,2}}{k_{2}\tau_{T_{2}}^{2}\left(s^{2}+\frac{2}{\tau_{T_{2}}}s+\frac{2}{\tau_{T_{2}}^{2}}\right)}\left[\left(s^{2}+\frac{2}{\tau_{q,2}}s+\frac{2}{\tau_{q,2}^{2}}\right)+\frac{2\omega_{b}c_{b}\left(s+\frac{\tau_{q,2}}{2}\right)}{\tau_{q,2}\rho_{2}c'_{p,2}}+\frac{2\omega_{b}c_{b}}{\tau_{q,2}^{2}\rho_{2}c'_{p,2}}\right]\tilde{T}_{b} \tag{60}
$$

$$
-\frac{\tau_{q,2}^{2}}{k_{2}\tau_{T_{2}}^{2}\left(s^{2}+\frac{2}{\tau_{T_{2}}}s+\frac{2}{\tau_{T_{2}}^{2}}\right)}\left[\begin{array}{l}\left(s^{2}+\frac{2}{\tau_{q,2}}s+\frac{2}{\tau_{q,2}^{2}}\right)\tilde{\tilde{q}}_{m}-\left(s+\frac{2}{\tau_{q,2}}\right)\tilde{q}_{b,0}\\\left(s^{2}+\frac{2}{\tau_{q,2}}s+\frac{2}{\tau_{q,2}^{2}}\right)\tilde{\tilde{q}}_{ext}-\left(s+\frac{2}{\tau_{q,2}}\right)\tilde{q}_{\text{ext},0}\end{array}\right]
$$

$$
\left(-\beta_{m3}^{2}\tilde{\tilde{T}}_{3}+\frac{d^{2}\tilde{\tilde{T}}_{3}}{dz^{2}}\right)-\frac{\left(\tau_{q,3}^{2}\rho_{3}c'_{p,3}\right)}{k_{3}\tau_{T_{3}}^{2}\left(s^{2}+\frac{2}{\tau_{T_{3}}}s+\frac{2}{\tau_{T_{3}}^{2}}\right)}\left(\begin{array}{l}s^{3}+\frac{2\left(\tau_{q,3}\rho_{3}c'_{p,3}+\omega_{b}c_{b}\frac{\tau_{q,3}^{2}}{2}\right)}{\tau_{q,3}^{2}\rho_{3}c'_{p,3}}s^{2}\\+\frac{2\left(\rho_{3}c'_{p,3}+\omega_{b}c_{b}\tau_{q,3}\right)}{\tau_{q,3}^{2}\rho_{3}c'_{p,3}}s+\frac{2\omega_{b}c_{b}}{\tau_{q,3}^{2}\rho_{3}c'_{p,3}}-\beta_{m3}^{2}\end{array}\right)\tilde{\tilde{T}}_{3}
$$

$$
=-\frac{\tau_{q,3}^{2}\rho_{3}c'_{p,3}}{k_{3}\tau_{T_{3}}^{2}\left(s^{2}+\frac{2}{\tau_{T_{3}}}s+\frac{2}{\tau_{T_{3}}^{2}}\right)}\left[\left(s^{2}+\frac{2}{\tau_{q,3}}s+\frac{2}{\tau_{q,3}^{2}}\right)+\frac{2\omega_{b}c_{b}\left(s+\frac{\tau_{q,3}}{2}\right)}{\tau_{q,3}\rho_{3}c'_{p,3}}+\frac{2\omega_{b}c_{b}}{\tau_{q,3}^{2}\rho_{3}c'_{p,3}}\right]\tilde{T}_{b} \tag{61}
$$

$$
-\frac{\tau_{q,3}^{2}}{k_{3}\tau_{T_{3}}^{2}\left(s^{2}+\frac{2}{\tau_{T_{3}}}s+\frac{2}{\tau_{T_{3}}^{2}}\right)}\left[\begin{array}{l}\left(s^{2}+\frac{2}{\tau_{q,3}}s+\frac{2}{\tau_{q,3}^{2}}\right)\tilde{\tilde{q}}_{m}-\left(s+\frac{2}{\tau_{q,3}}\right)\tilde{q}_{b,0}\\\left(s^{2}+\frac{2}{\tau_{q,3}}s+\frac{2}{\tau_{q,3}^{2}}\right)\tilde{\tilde{q}}_{ext}-\left(s+\frac{2}{\tau_{q,3}}\right)\tilde{q}_{\text{ext},0}\end{array}\right]
$$

Rearranging the above models

$$\frac{d^2\tilde{\tilde{T}}_1}{dz^2}-\frac{\left(\tau_{q,1}^2\rho_1 c'_{p,1}\right)}{k_1\tau_{T_1}^2\left(s^2+\frac{2}{\tau_{T_1}}s+\frac{2}{\tau_{T_1}^2}\right)}\left(\begin{array}{l}s^3+\frac{2\left(\tau_{q,1}\rho_1 c'_{p,1}+\omega_b c_b\frac{\tau_{q,1}^2}{2}\right)}{\tau_{q,1}^2\rho_1 c'_{p,1}}s^2\\+\frac{2\left(\rho_1 c_{p,1}+\omega_b c_b\tau_{q,1}\right)}{\tau_{q,1}^2\rho_1 c'_{p,1}}s+\frac{2\omega_b c_b}{\tau_{q,1}^2\rho_1 c'_{p,1}}-\beta_{m1}^2\end{array}\right)\tilde{\tilde{T}}_1$$

$$=-\frac{\tau_{q,1}^2\rho_1 c'_{p,1}}{k_1\tau_{T_1}^2\left(s^2+\frac{2}{\tau_{T_1}}s+\frac{2}{\tau_{T_1}^2}\right)}\left[\left(s^2+\frac{2}{\tau_{q,1}}s+\frac{2}{\tau_{q,1}^2}\right)+\frac{2\omega_b c_b\left(s+\frac{\tau_{q,1}}{2}\right)}{\tau_{q,1}\rho_1 c'_{p,1}}+\frac{2\omega_b c_b}{\tau_{q,1}^2\rho_1 c'_{p,1}}\right]\tilde{T}_b \tag{62}$$

$$-\frac{\tau_{q,1}^2}{k_1\tau_{T_1}^2\left(s^2+\frac{2}{\tau_{T_1}}s+\frac{2}{\tau_{T_1}^2}\right)}\left[\begin{array}{l}\left(s^2+\frac{2}{\tau_{q,1}}s+\frac{2}{\tau_{q,1}^2}\right)\tilde{\tilde{q}}_m-\left(s+\frac{2}{\tau_{q,1}}\right)\tilde{q}_{b,0}\\\left(s^2+\frac{2}{\tau_{q,1}}s+\frac{2}{\tau_{q,1}^2}\right)\tilde{\tilde{q}}_{ext}-\left(s+\frac{2}{\tau_{q,1}}\right)\tilde{q}_{\mathrm{ext},0}\end{array}\right]$$

$$\frac{d^2\tilde{\tilde{T}}_2}{dz^2}-\frac{\left(\tau_{q,2}^2\rho_2 c'_{p,2}\right)}{k_2\tau_{T_2}^2\left(s^2+\frac{2}{\tau_{T_2}}s+\frac{2}{\tau_{T_2}^2}\right)}\left(\begin{array}{l}s^3+\frac{2\left(\tau_{q,2}\rho_2 c'_{p,2}+\omega_b c_b\frac{\tau_{q,2}^2}{2}\right)}{\tau_{q,2}^2\rho_2 c'_{p,2}}s^2\\+\frac{2\left(\rho_2 c'_{p,2}+\omega_b c_b\tau_{q,2}\right)}{\tau_{q,2}^2\rho_2 c'_{p,2}}s+\frac{2\omega_b c_b}{\tau_{q,2}^2\rho_2 c'_{p,2}}-\beta_{m2}^2\end{array}\right)\tilde{\tilde{T}}_2$$

$$=-\frac{\tau_{q,2}^2\rho_2 c'_{p,2}}{k_2\tau_{T_2}^2\left(s^2+\frac{2}{\tau_{T_2}}s+\frac{2}{\tau_{T_2}^2}\right)}\left[\left(\left(\left(s^2+\frac{2}{\tau_{q,2}}s+\frac{2}{\tau_{q,2}^2}\right)\right)\right)+\frac{2\omega_b c_b\left(s+\frac{\tau_{q,2}}{2}\right)}{\tau_{q,2}\rho_2 c'_{p,2}}+\frac{2\omega_b c_b}{\tau_{q,2}^2\rho_2 c'_{p,2}}\right]\tilde{T}_b \tag{63}$$

$$-\frac{\tau_{q,2}^2}{k_2\tau_{T_2}^2\left(s^2+\frac{2}{\tau_{T_2}}s+\frac{2}{\tau_{T_2}^2}\right)}\left[\begin{array}{l}\left(s^2+\frac{2}{\tau_{q,2}}s+\frac{2}{\tau_{q,2}^2}\right)\tilde{\tilde{q}}_m-\left(s+\frac{2}{\tau_{q,2}}\right)\tilde{q}_{b,0}\\\left(s^2+\frac{2}{\tau_{q,2}}s+\frac{2}{\tau_{q,2}^2}\right)\tilde{\tilde{q}}_{ext}-\left(s+\frac{2}{\tau_{q,2}}\right)\tilde{q}_{\mathrm{ext},0}\end{array}\right]$$

$$\frac{d^2\tilde{\tilde{T}}_3}{dz^2}-\frac{\left(\tau_{q,3}^2\rho_3c'_{p,3}\right)}{k_3\tau_{T_3}^2\left(s^2+\frac{2}{\tau_{T_3}}s+\frac{2}{\tau_{T_3}^2}\right)}\left(\begin{array}{l}s^3+\frac{2\left(\tau_{q,3}\rho_3c'_{p,3}+\omega_bc_b\frac{\tau_{q,3}^2}{2}\right)}{\tau_{q,3}^2\rho_3c'_{p,3}}s^2\\+\frac{2\left(\rho_3c'_{p,3}+\omega_bc_b\tau_{q,3}\right)}{\tau_{q,3}^2\rho_3c'_{p,3}}s+\frac{2\omega_bc_b}{\tau_{q,3}^2\rho_3c'_{p,3}}-\beta_{m3}^2\end{array}\right)\tilde{\tilde{T}}_3$$

$$=-\frac{\tau_{q,3}^2\rho_3c'_{p,3}}{k_3\tau_{T_3}^2\left(s^2+\frac{2}{\tau_{T_3}}s+\frac{2}{\tau_{T_3}^2}\right)}\left[\left(\left(s^2+\frac{2}{\tau_{q,3}}s+\frac{2}{\tau_{q,3}^2}\right)\right)+\frac{2\omega_bc_b\left(s+\frac{\tau_{q,3}}{2}\right)}{\tau_{q,3}\rho_3c'_{p,3}}+\frac{2\omega_bc_b}{\tau_{q,3}^2\rho_3c'_{p,3}}\right]\tilde{T}_b \tag{64}$$

$$-\frac{\tau_{q,3}^2}{k_3\tau_{T_3}^2\left(s^2+\frac{2}{\tau_{T_3}}s+\frac{2}{\tau_{T_3}^2}\right)}\left[\begin{array}{l}\left(s^2+\frac{2}{\tau_{q,3}}s+\frac{2}{\tau_{q,3}^2}\right)\tilde{\tilde{q}}_m-\left(s+\frac{2}{\tau_{q,3}}\right)\tilde{q}_{b,0}\\\left(s^2+\frac{2}{\tau_{q,3}}s+\frac{2}{\tau_{q,3}^2}\right)\tilde{\tilde{q}}_{ext}-\left(s+\frac{2}{\tau_{q,3}}\right)\tilde{q}_{\text{ext},0}\end{array}\right]$$

**4.4.1. The boundary conditions in Laplace and Fourier domains are**

$$s>0,\quad z=0,\ 0\le x\le L\ \ -k\frac{\partial\tilde{\tilde{T}}_1}{\partial z}=h\left(\frac{\tilde{T}_a}{s}-\tilde{\tilde{T}}_1\right) \tag{65a}$$

$$s>0,\quad z=H,\ 0\le x\le L,\ \tilde{\tilde{T}}_3=\frac{T_b}{s} \tag{65b}$$

**4.4.2. Interlayer conditions in Laplace domain are:**

$$s>0,\ \ z=z_1,\quad 0\le x\le L,\quad \tilde{\tilde{T}}_1=\tilde{\tilde{T}}_2, \tag{66a}$$

$$s>0,\ \ z=z_1,\quad 0\le x\le L,\quad k_1\frac{d\tilde{\tilde{T}}_1}{dz}=k_2\frac{d\tilde{\tilde{T}}_2}{dz}, \tag{66b}$$

$$s>0,\ \ z=z_2,\quad 0\le x\le L,\quad \tilde{\tilde{T}}_2=\tilde{\tilde{T}}_3, \tag{66c}$$

$$s>0,\ \ z=z_2,\quad 0\le x\le L,\quad k_2\frac{d\tilde{\tilde{T}}_2}{dz}=k_3\frac{d\tilde{\tilde{T}}_3}{dz}, \tag{66d}$$

The above Eqs. (62) - (64) can be written as

$$
\frac{d^2\tilde{\bar{T}}_1}{dz^2}-\frac{\alpha_1\left(s^3+\chi_1 s^2+\delta_1 s+\lambda_1\right)}{\left(s^2+2\tau_{T_1}^{-1}s+2\tau_{T_1}^{-2}\right)}\tilde{\bar{T}}_1
$$

$$
=-\frac{\mu_1}{\left(s^2+2\tau_{T_1}^{-1}s+2\tau_{T_1}^{-2}\right)}\left[\left(s^2+2\tau_{q_1}^{-1}s+2\tau_{q_1}^{-2}\right)+\xi_1\left(s+2^{-1}\tau_{q,1}\right)+\zeta_1\right]\tilde{T}_b \tag{67}
$$

$$
-\frac{\eta_1}{\left(s^2+2\tau_{T_1}^{-1}s+2\tau_{T_1}^{-2}\right)}\begin{bmatrix}\left(s^2+2\tau_{q_1}^{-1}s+2\tau_{q_1}^{-2}\right)\tilde{\bar{q}}_m-\left(s+2^{-1}\tau_{q,1}\right)\tilde{q}_{b,0}\\ \left(s^2+2\tau_{q_1}^{-1}s+2\tau_{q_1}^{-2}\right)\tilde{\bar{q}}_{ext}-\left(s+2^{-1}\tau_{q,1}\right)\tilde{q}_{\mathrm{ext},0}\end{bmatrix}
$$

$$
\frac{d^2\tilde{\bar{T}}_2}{dz^2}-\frac{\alpha_2\left(s^3+\chi_2 s^2+\delta_2 s+\lambda_2\right)}{\left(s^2+2\tau_{T_2}^{-1}s+2\tau_{T_2}^{-2}\right)}\tilde{\bar{T}}_2
$$

$$
=-\frac{\mu_2}{\left(s^2+2\tau_{T_2}^{-1}s+2\tau_{T_2}^{-2}\right)}\left[\left(s^2+2\tau_{q_2}^{-1}s+2\tau_{q_2}^{-2}\right)+\xi_2\left(s+2^{-1}\tau_{q_2}\right)+\zeta_2\right]\tilde{T}_b \tag{68}
$$

$$
-\frac{\eta_2}{\left(s^2+2\tau_{T_2}^{-1}s+2\tau_{T_2}^{-2}\right)}\begin{bmatrix}\left(s^2+2\tau_{q_2}^{-1}s+2\tau_{q_2}^{-2}\right)\tilde{\bar{q}}_m-\left(s+2^{-1}\tau_{q_2}\right)\tilde{q}_{b,0}\\ \left(s^2+2\tau_{q_2}^{-1}s+2\tau_{q_2}^{-2}\right)\tilde{\bar{q}}_{ext}-\left(s+2^{-1}\tau_{q_2}\right)\tilde{q}_{\mathrm{ext},0}\end{bmatrix}
$$

$$
\frac{d^2\tilde{\bar{T}}_3}{dz^2}-\frac{\alpha_3\left(s^3+\chi_3 s^2+\delta_3 s+\lambda_3\right)}{\left(s^2+2\tau_{T_3}^{-1}s+2\tau_{T_3}^{-2}\right)}\tilde{\bar{T}}_3
$$

$$
=-\frac{\mu_3}{\left(s^2+2\tau_{T_3}^{-1}s+2\tau_{T_3}^{-2}\right)}\left[\left(s^2+2\tau_{q_3}^{-1}s+2\tau_{q_3}^{-2}\right)+\xi_3\left(s+2^{-1}\tau_{q_3}\right)+\zeta_3\right]\tilde{T}_b \tag{69}
$$

$$
-\frac{\eta_3}{\left(s^2+2\tau_{T_3}^{-1}s+2\tau_{T_3}^{-2}\right)}\begin{bmatrix}\left(s^2+2\tau_{q_3}^{-1}s+2\tau_{q_3}^{-2}\right)\tilde{\bar{q}}_m-\left(s+2^{-1}\tau_{q_3}\right)\tilde{q}_{b,0}\\ \left(s^2+2\tau_{q_3}^{-1}s+2\tau_{q_3}^{-2}\right)\tilde{\bar{q}}_{ext}-\left(s+2^{-1}\tau_{q_3}\right)\tilde{q}_{\mathrm{ext},0}\end{bmatrix}
$$

where

$$
\alpha_1=\frac{\left(\tau_{q,1}^2\rho_1 c'_{p,1}\right)}{k_1\tau_{T_1}^2},\quad \chi_1=\frac{2\left(\tau_{q,1}\rho_1 c'_{p,1}+\omega_b c_b\,\frac{\tau_{q,1}^2}{2}\right)}{\tau_{q,1}^2\rho_1 c'_{p,1}},\quad \delta_1=\frac{2\left(\rho_1 c'_{p,1}+\omega_b c_b\tau_{q,1}\right)}{\tau_{q,1}^2\rho_1 c'_{p,1}},
$$

$$
\lambda_1=\frac{2\omega_b c_b}{\tau_{q,1}^2\rho_1 c'_{p,1}}-\beta_{m1}^2,\quad \mu_1=\frac{\tau_{q,1}^2\rho_1 c'_{p,1}}{k_1\tau_{T_1}^2},\quad \xi_1=\frac{2\omega_b c_b}{\tau_{q,1}\rho_1 c'_{p,1}},\quad \zeta_1=\frac{2\omega_b c_b}{\tau_{q,1}^2\rho_1 c'_{p,1}},\quad \eta_1=\frac{\tau_{q,1}^2}{k_1\tau_{T_1}^2}
$$

$$
\alpha_2=\frac{-\left(\tau_{q,2}^2\rho_2 c'_{p,2}\right)}{k_2\tau_{T_2}^2},\quad \chi_2=\frac{2\left(\tau_{q,2}\rho_2 c'_{p,2}+\omega_b c_b\,\frac{\tau_{q,2}^2}{2}\right)}{\tau_{q,2}^2\rho_2 c'_{p,2}},\quad \delta_2=\frac{2\left(\rho_2 c'_{p,2}+\omega_b c_b\tau_{q,2}\right)}{\tau_{q,2}^2\rho_2 c'_{p,2}},
$$

$$
\lambda_2=\frac{2\omega_b c_b}{\tau_{q,2}^2\rho_2 c'_{p,2}}-\beta_{m2}^2,\quad \xi_2=\frac{2\omega_b c_b}{\tau_{q,2}\rho_2 c'_{p,2}},\quad \zeta_2=\frac{2\omega_b c_b}{\tau_{q,2}^2\rho_2 c'_{p,2}},\quad \eta_2=\frac{\tau_{q,2}^2}{k_2\tau_{T_2}^2}
$$

$$\alpha_3 = \frac{-\left(\tau_{q,3}^2\rho_3 c'_{p,3}\right)}{k_3\tau_{T_3}^2}, \quad \chi_3 = \frac{2\left(\tau_{q,3}\rho_3 c'_{p,3} + \omega_b c_b \frac{\tau_{q,3}^2}{2}\right)}{\tau_{q,3}^2\rho_3 c'_{p,3}}, \quad \delta_3 = \frac{2\left(\rho_3 c'_{p,3} + \omega_b c_b \tau_{q,3}\right)}{\tau_{q,3}^2\rho_3 c'_{p,3}},$$

$$\lambda_3 = \frac{2\omega_b c_b}{\tau_{q,3}^2\rho_3 c'_{p,3}} - \beta_{m3}^2, \quad \xi_3 = \frac{2\omega_b c_b}{\tau_{q,3}\rho_3 c'_{p,3}}, \quad \zeta_3 = \frac{2\omega_b c_b}{\tau_{q,3}^2\rho_3 c'_{p,3}}, \quad \eta_3 = \frac{\tau_{q,3}^2}{k_3\tau_{T_3}^2}$$

The solution of the above Eqs. (67) – (69) are

$$\begin{aligned}\tilde{\tilde{T}}_1 &= A_1 cosh\left(\sqrt{\frac{\alpha_1\left(s^3+\chi_1 s^2+\delta_1 s+\lambda_1\right)}{\left(s^2+2\tau_{T_1}^{-1}s+2\tau_{T_1}^{-2}\right)}}\right)z + B_1 sinh\left(\sqrt{\frac{\alpha_1\left(s^3+\chi_1 s^2+\delta_1 s+\lambda_1\right)}{\left(s^2+2\tau_{T_1}^{-1}s+2\tau_{T_1}^{-2}\right)}}\right)z \\ &+ \frac{\mu_1}{\alpha_1\left(s^3+\chi_1 s^2+\delta_1 s+\lambda_1\right)}\left[\left(s^2+2\tau_{q_1}^{-1}s+2\tau_{q_1}^{-2}\right)+\xi_1\left(s+2^{-1}\tau_{q,1}\right)+\zeta_1\right]\tilde{T}_b \\ &+ \frac{\eta_1}{\alpha_1\left(s^3+\chi_1 s^2+\delta_1 s+\lambda_1\right)}\begin{bmatrix}\left(s^2+2\tau_{q_1}^{-1}s+2\tau_{q_1}^{-2}\right)\tilde{\tilde{q}}_m-\left(s+2^{-1}\tau_{q,1}\right)\tilde{q}_{b,0} \\ \left(s^2+2\tau_{q_1}^{-1}s+2\tau_{q_1}^{-2}\right)\tilde{\tilde{q}}_{ext}-\left(s+2^{-1}\tau_{q,1}\right)\tilde{q}_{\text{ext},0}\end{bmatrix}\end{aligned} \tag{70}$$

$$\begin{aligned}\tilde{\tilde{T}}_2 &= A_2 cosh\left(\sqrt{\frac{\alpha_2\left(s^3+\chi_2 s^2+\delta_2 s+\lambda_2\right)}{\left(s^2+2\tau_{T_2}^{-1}s+2\tau_{T_2}^{-2}\right)}}\right)z + B_2 sinh\left(\sqrt{\frac{\alpha_2\left(s^3+\chi_2 s^2+\delta_2 s+\lambda_2\right)}{\left(s^2+2\tau_{T_2}^{-1}s+2\tau_{T_2}^{-2}\right)}}\right)z \\ &+ \frac{\mu_2}{\alpha_2\left(s^3+\chi_2 s^2+\delta_2 s+\lambda_2\right)}\left[\left(s^2+2\tau_{q_2}^{-1}s+2\tau_{q_2}^{-2}\right)+\xi_2\left(s+2^{-1}\tau_{q_2}\right)+\zeta_2\right]\tilde{T}_b \\ &+ \frac{\eta_2}{\alpha_2\left(s^3+\chi_2 s^2+\delta_2 s+\lambda_2\right)}\begin{bmatrix}\left(s^2+2\tau_{q_2}^{-1}s+2\tau_{q_2}^{-2}\right)\tilde{\tilde{q}}_m-\left(s+2^{-1}\tau_{q_2}\right)\tilde{q}_{b,0} \\ \left(s^2+2\tau_{q_2}^{-1}s+2\tau_{q_2}^{-2}\right)\tilde{\tilde{q}}_{ext}-\left(s+2^{-1}\tau_{q_2}\right)\tilde{q}_{\text{ext},0}\end{bmatrix}\end{aligned} \tag{71}$$

$$\begin{aligned}\tilde{\tilde{T}}_3 &= A_3 cosh\left(\sqrt{\frac{\alpha_3\left(s^3+\chi_3 s^2+\delta_3 s+\lambda_3\right)}{\left(s^2+2\tau_{T_3}^{-1}s+2\tau_{T_3}^{-2}\right)}}\right)z + B_3\left(sinh\sqrt{\frac{\alpha_3\left(s^3+\chi_3 s^2+\delta_3 s+\lambda_3\right)}{\left(s^2+2\tau_{T_3}^{-1}s+2\tau_{T_3}^{-2}\right)}}\right)z \\ &+ \frac{\mu_3}{\alpha_3\left(s^3+\chi_3 s^2+\delta_3 s+\lambda_3\right)}\left[\left(s^2+2\tau_{q_3}^{-1}s+2\tau_{q_3}^{-2}\right)+\xi_3\left(s+2^{-1}\tau_{q_3}\right)+\zeta_3\right]\tilde{T}_b \\ &+ \frac{\eta_2}{\alpha_3\left(s^3+\chi_3 s^2+\delta_3 s+\lambda_3\right)}\begin{bmatrix}\left(s^2+2\tau_{q_3}^{-1}s+2\tau_{q_3}^{-2}\right)\tilde{\tilde{q}}_m-\left(s+2^{-1}\tau_{q_3}\right)\tilde{q}_{b,0} \\ \left(s^2+2\tau_{q_3}^{-1}s+2\tau_{q_3}^{-2}\right)\tilde{\tilde{q}}_{ext}-\left(s+2^{-1}\tau_{q_3}\right)\tilde{q}_{\text{ext},0}\end{bmatrix}\end{aligned} \tag{72}$$

Applying the boundary conditions in Eqs. (65a) and Eqs. (65b) and the intermediate conditions in (66a-d), we arrived at the following equations for the unknown coefficients

$$
hA_1-\left(\sqrt{\frac{\alpha_1\left(s^3+\chi_1 s^2+\delta_1 s+\lambda_1\right)}{\left(s^2+2\tau_{T_1}^{-1}s+2\tau_{T_1}^{-2}\right)}}\right)kB_1=h\left\{\frac{\tilde{T}_a}{s}-\left\{\begin{array}{l}\dfrac{\mu_1}{\alpha_1\left(s^3+\chi_1 s^2+\delta_1 s+\lambda_1\right)}\begin{bmatrix}\left(s^2+2\tau_{q_1}^{-1}s+2\tau_{q_1}^{-2}\right)\\+\xi_1\left(s+2^{-1}\tau_{q,1}\right)+\zeta_1\end{bmatrix}\tilde{T}_b\\+\dfrac{\eta_1}{\alpha_1\left(s^3+\chi_1 s^2+\delta_1 s+\lambda_1\right)}\begin{bmatrix}\left(s^2+2\tau_{q_1}^{-1}s+2\tau_{q_1}^{-2}\right)\tilde{\tilde{q}}_m\\-\left(s+2^{-1}\tau_{q,1}\right)\tilde{q}_{b,0}\\\left(s^2+2\tau_{q_1}^{-1}s+2\tau_{q_1}^{-2}\right)\tilde{\tilde{q}}_{ext}\\-\left(s+2^{-1}\tau_{q,1}\right)\tilde{q}_{\mathrm{ext},0}\end{bmatrix}\end{array}\right\}\right\} \tag{73a}
$$

$$
\begin{aligned}
&A_1 cosh\left(\sqrt{\frac{\alpha_1\left(s^3+\chi_1 s^2+\delta_1 s+\lambda_1\right)}{\left(s^2+2\tau_{T_1}^{-1}s+2\tau_{T_1}^{-2}\right)}}\right)z_1+B_1 sinh\left(\sqrt{\frac{\alpha_1\left(s^3+\chi_1 s^2+\delta_1 s+\lambda_1\right)}{\left(s^2+2\tau_{T_1}^{-1}s+2\tau_{T_1}^{-2}\right)}}\right)z_1\\
&-A_2 cosh\left(\sqrt{\frac{\alpha_2\left(s^3+\chi_2 s^2+\delta_2 s+\lambda_2\right)}{\left(s^2+2\tau_{T_2}^{-1}s+2\tau_{T_2}^{-2}\right)}}\right)z_1-B_2 sinh\left(\sqrt{\frac{\alpha_2\left(s^3+\chi_2 s^2+\delta_2 s+\lambda_2\right)}{\left(s^2+2\tau_{T_2}^{-1}s+2\tau_{T_2}^{-2}\right)}}\right)z_1\\
&\quad=\frac{\mu_2}{\alpha_2\left(s^3+\chi_2 s^2+\delta_2 s+\lambda_2\right)}\left[\left(s^2+2\tau_{q_2}^{-1}s+2\tau_{q_2}^{-2}\right)+\xi_2\left(s+2^{-1}\tau_{q_2}\right)+\zeta_2\right]\tilde{T}_b\\
&\quad+\frac{\eta_2}{\alpha_2\left(s^3+\chi_2 s^2+\delta_2 s+\lambda_2\right)}\begin{bmatrix}\left(s^2+2\tau_{q_2}^{-1}s+2\tau_{q_2}^{-2}\right)\tilde{\tilde{q}}_m-\left(s+2^{-1}\tau_{q_2}\right)\tilde{q}_{b,0}\\\left(s^2+2\tau_{q_2}^{-1}s+2\tau_{q_2}^{-2}\right)\tilde{\tilde{q}}_{ext}-\left(s+2^{-1}\tau_{q_2}\right)\tilde{q}_{\mathrm{ext},0}\end{bmatrix}\\
&\quad-\frac{\mu_1}{\alpha_1\left(s^3+\chi_1 s^2+\delta_1 s+\lambda_1\right)}\left[\left(s^2+2\tau_{q_1}^{-1}s+2\tau_{q_1}^{-2}\right)+\xi_1\left(s+2^{-1}\tau_{q,1}\right)+\zeta_1\right]\tilde{T}_b\\
&\quad-\frac{\eta_1}{\alpha_1\left(s^3+\chi_1 s^2+\delta_1 s+\lambda_1\right)}\begin{bmatrix}\left(s^2+2\tau_{q_1}^{-1}s+2\tau_{q_1}^{-2}\right)\tilde{\tilde{q}}_m-\left(s+2^{-1}\tau_{q,1}\right)\tilde{q}_{b,0}\\\left(s^2+2\tau_{q_1}^{-1}s+2\tau_{q_1}^{-2}\right)\tilde{\tilde{q}}_{ext}-\left(s+2^{-1}\tau_{q,1}\right)\tilde{q}_{\mathrm{ext},0}\end{bmatrix}
\end{aligned} \tag{73b}
$$

$$
\begin{aligned}
&A_1\left(\sqrt{\frac{\alpha_1\left(s^3+\chi_1 s^2+\delta_1 s+\lambda_1\right)}{\left(s^2+2\tau_{T_1}^{-1}s+2\tau_{T_1}^{-2}\right)}}\right)sinh\left(\sqrt{\frac{\alpha_1\left(s^3+\chi_1 s^2+\delta_1 s+\lambda_1\right)}{\left(s^2+2\tau_{T_1}^{-1}s+2\tau_{T_1}^{-2}\right)}}\right)z_1\\
&+B_1\left(\sqrt{\frac{\alpha_1\left(s^3+\chi_1 s^2+\delta_1 s+\lambda_1\right)}{\left(s^2+2\tau_{T_1}^{-1}s+2\tau_{T_1}^{-2}\right)}}\right)cosh\left(\sqrt{\frac{\alpha_1\left(s^3+\chi_1 s^2+\delta_1 s+\lambda_1\right)}{\left(s^2+2\tau_{T_1}^{-1}s+2\tau_{T_1}^{-2}\right)}}\right)z_1\\
&-A_2\left(\sqrt{\frac{\alpha_2\left(s^3+\chi_2 s^2+\delta_2 s+\lambda_2\right)}{\left(s^2+2\tau_{T_2}^{-1}s+2\tau_{T_2}^{-2}\right)}}\right)sinh\left(\sqrt{\frac{\alpha_2\left(s^3+\chi_2 s^2+\delta_2 s+\lambda_2\right)}{\left(s^2+2\tau_{T_2}^{-1}s+2\tau_{T_2}^{-2}\right)}}\right)z_1\\
&-B_2\left(\sqrt{\frac{\alpha_2\left(s^3+\chi_2 s^2+\delta_2 s+\lambda_2\right)}{\left(s^2+2\tau_{T_2}^{-1}s+2\tau_{T_2}^{-2}\right)}}\right)cosh\left(\sqrt{\frac{\alpha_2\left(s^3+\chi_2 s^2+\delta_2 s+\lambda_2\right)}{\left(s^2+2\tau_{T_2}^{-1}s+2\tau_{T_2}^{-2}\right)}}\right)z_1=0
\end{aligned} \tag{73c}
$$

$$A_2 cosh\left(\sqrt{\frac{\alpha_2\left(s^3+\chi_2 s^2+\delta_2 s+\lambda_2\right)}{\left(s^2+2\tau_{T_2}^{-1}s+2\tau_{T_2}^{-2}\right)}}\right)z_2+B_2 sinh\left(\sqrt{\frac{\alpha_2\left(s^3+\chi_2 s^2+\delta_2 s+\lambda_2\right)}{\left(s^2+2\tau_{T_2}^{-1}s+2\tau_{T_2}^{-2}\right)}}\right)z_2$$
$$-A_3 cosh\left(\sqrt{\frac{\alpha_3\left(s^3+\chi_3 s^2+\delta_3 s+\lambda_3\right)}{\left(s^2+2\tau_{T_3}^{-1}s+2\tau_{T_3}^{-2}\right)}}\right)z_2-B_3 sinh\left(\sqrt{\frac{\alpha_3\left(s^3+\chi_3 s^2+\delta_3 s+\lambda_3\right)}{\left(s^2+2\tau_{T_3}^{-1}s+2\tau_{T_3}^{-2}\right)}}\right)z_2$$
$$=\frac{\mu_3}{\alpha_3\left(s^3+\chi_3 s^2+\delta_3 s+\lambda_3\right)}\left[\left(s^2+2\tau_{q_3}^{-1}s+2\tau_{q_3}^{-2}\right)+\xi_3\left(s+2^{-1}\tau_{q_3}\right)+\zeta_3\right]\tilde{T}_b$$
$$+\frac{\eta_2}{\alpha_3\left(s^3+\chi_3 s^2+\delta_3 s+\lambda_3\right)}\begin{bmatrix}\left(s^2+2\tau_{q_3}^{-1}s+2\tau_{q_3}^{-2}\right)\tilde{\bar{q}}_m-\left(s+2^{-1}\tau_{q_3}\right)\tilde{q}_{b,0}\\ \left(s^2+2\tau_{q_3}^{-1}s+2\tau_{q_3}^{-2}\right)\tilde{\bar{q}}_{ext}-\left(s+2^{-1}\tau_{q_3}\right)\tilde{q}_{\text{ext},0}\end{bmatrix}$$
$$-\frac{\mu_2}{\alpha_2\left(s^3+\chi_2 s^2+\delta_2 s+\lambda_2\right)}\left[\left(s^2+2\tau_{q_2}^{-1}s+2\tau_{q_2}^{-2}\right)+\xi_2\left(s+2^{-1}\tau_{q_2}\right)+\zeta_2\right]\tilde{T}_b$$
$$-\frac{\eta_2}{\alpha_2\left(s^3+\chi_2 s^2+\delta_2 s+\lambda_2\right)}\begin{bmatrix}\left(s^2+2\tau_{q_2}^{-1}s+2\tau_{q_2}^{-2}\right)\tilde{\bar{q}}_m-\left(s+2^{-1}\tau_{q_2}\right)\tilde{q}_{b,0}\\ \left(s^2+2\tau_{q_2}^{-1}s+2\tau_{q_2}^{-2}\right)\tilde{\bar{q}}_{ext}-\left(s+2^{-1}\tau_{q_2}\right)\tilde{q}_{\text{ext},0}\end{bmatrix} \tag{73d}$$

$$A_2\left(\sqrt{\frac{\alpha_2\left(s^3+\chi_2 s^2+\delta_2 s+\lambda_2\right)}{\left(s^2+2\tau_{T_2}^{-1}s+2\tau_{T_2}^{-2}\right)}}\right)sinh\left(\sqrt{\frac{\alpha_2\left(s^3+\chi_2 s^2+\delta_2 s+\lambda_2\right)}{\left(s^2+2\tau_{T_2}^{-1}s+2\tau_{T_2}^{-2}\right)}}\right)z_2$$
$$+B_2\left(\sqrt{\frac{\alpha_2\left(s^3+\chi_2 s^2+\delta_2 s+\lambda_2\right)}{\left(s^2+2\tau_{T_2}^{-1}s+2\tau_{T_2}^{-2}\right)}}\right)cosh\left(\sqrt{\frac{\alpha_2\left(s^3+\chi_2 s^2+\delta_2 s+\lambda_2\right)}{\left(s^2+2\tau_{T_2}^{-1}s+2\tau_{T_2}^{-2}\right)}}\right)z_2 \tag{73e}$$
$$-A_3\left(\sqrt{\frac{\alpha_3\left(s^3+\chi_3 s^2+\delta_3 s+\lambda_3\right)}{\left(s^2+2\tau_{T_3}^{-1}s+2\tau_{T_3}^{-2}\right)}}\right)sinh\left(\sqrt{\frac{\alpha_3\left(s^3+\chi_3 s^2+\delta_3 s+\lambda_3\right)}{\left(s^2+2\tau_{T_3}^{-1}s+2\tau_{T_3}^{-2}\right)}}\right)z_2$$
$$-B_3\left(\sqrt{\frac{\alpha_3\left(s^3+\chi_3 s^2+\delta_3 s+\lambda_3\right)}{\left(s^2+2\tau_{T_3}^{-1}s+2\tau_{T_3}^{-2}\right)}}\right)cosh\left(\sqrt{\frac{\alpha_3\left(s^3+\chi_3 s^2+\delta_3 s+\lambda_3\right)}{\left(s^2+2\tau_{T_3}^{-1}s+2\tau_{T_3}^{-2}\right)}}\right)z_2=0$$

$$A_3 cosh\left(\sqrt{\frac{\alpha_3\left(s^3+\chi_3 s^2+\delta_3 s+\lambda_3\right)}{\left(s^2+2\tau_{T_3}^{-1}s+2\tau_{T_3}^{-2}\right)}}\right)H+B_3\left(sinh\sqrt{\frac{\alpha_3\left(s^3+\chi_3 s^2+\delta_3 s+\lambda_3\right)}{\left(s^2+2\tau_{T_3}^{-1}s+2\tau_{T_3}^{-2}\right)}}\right)H$$
$$+\frac{\mu_3}{\alpha_3\left(s^3+\chi_3 s^2+\delta_3 s+\lambda_3\right)}\left[\left(s^2+2\tau_{q_3}^{-1}s+2\tau_{q_3}^{-2}\right)+\xi_3\left(s+2^{-1}\tau_{q_3}\right)+\zeta_3\right]\tilde{T}_b \tag{73f}$$
$$+\frac{\eta_2}{\alpha_3\left(s^3+\chi_3 s^2+\delta_3 s+\lambda_3\right)}\begin{bmatrix}\left(s^2+2\tau_{q_3}^{-1}s+2\tau_{q_3}^{-2}\right)\tilde{\bar{q}}_m-\left(s+2^{-1}\tau_{q_3}\right)\tilde{q}_{b,0}\\ \left(s^2+2\tau_{q_3}^{-1}s+2\tau_{q_3}^{-2}\right)\tilde{\bar{q}}_{ext}-\left(s+2^{-1}\tau_{q_3}\right)\tilde{q}_{\text{ext},0}\end{bmatrix}=\frac{\tilde{T}_b}{s}$$

The above Eqs. (73a) - (73f) can be written in matrix form as:

$$\begin{bmatrix} h & \gamma_{12}(s) & 0 & 0 & 0 & 0 \\ \gamma_{21}(s) & \gamma_{22}(s) & \gamma_{23}(s) & \gamma_{24}(s) & 0 & 0 \\ \gamma_{31}(s) & \gamma_{32}(s) & \gamma_{33}(s) & \gamma_{34}(s) & 0 & 0 \\ 0 & 0 & \gamma_{43}(s) & \gamma_{44}(s) & \gamma_{45}(s) & \gamma_{46}(s) \\ 0 & 0 & \gamma_{53}(s) & \gamma_{54}(s) & \gamma_{55}(s) & \gamma_{56}(s) \\ 0 & 0 & 0 & 0 & \gamma_{65}(s) & \gamma_{66}(s) \end{bmatrix}\begin{bmatrix} A_1 \\ B_1 \\ A_2 \\ B_2 \\ A_3 \\ B_3 \end{bmatrix}=\begin{bmatrix} \Omega_1(s,\beta_{m1}) \\ \Omega_2(s) \\ 0 \\ \Omega_4(s) \\ 0 \\ \Omega_6(s) \end{bmatrix} \tag{74}$$

where

$$\gamma_{12}(s)=-\left(\sqrt{\frac{\alpha_1\left(s^3+\chi_1 s^2+\delta_1 s+\lambda_1\right)}{\left(s^2+2\tau_{T_1}^{-1}s+2\tau_{T_1}^{-2}\right)}}\right)k,$$

$$\Omega_1(s,\beta_{m1})=h\left\{\frac{\tilde{T}_a}{s}-\left\{\begin{array}{l}\dfrac{\mu_1}{\alpha_1\left(s^3+\chi_1 s^2+\delta_1 s+\lambda_1\right)}\left[\begin{array}{l}\left(s^2+2\tau_{q_1}^{-1}s+2\tau_{q_1}^{-2}\right)\\+\xi_1\left(s+2^{-1}\tau_{q,1}\right)+\zeta_1\end{array}\right]\tilde{T}_b\\+\dfrac{\eta_1}{\alpha_1\left(s^3+\chi_1 s^2+\delta_1 s+\lambda_1\right)}\left[\begin{array}{l}\left(s^2+2\tau_{q_1}^{-1}s+2\tau_{q_1}^{-2}\right)\tilde{\tilde{q}}_m\\-\left(s+2^{-1}\tau_{q,1}\right)\tilde{q}_{b,0}\\\left(s^2+2\tau_{q_1}^{-1}s+2\tau_{q_1}^{-2}\right)\tilde{\tilde{q}}_{ext}\\-\left(s+2^{-1}\tau_{q,1}\right)\tilde{q}_{\text{ext},0}\end{array}\right]\end{array}\right\}\right\}$$

$$\gamma_{21}(s)=cosh\left(\sqrt{\frac{\alpha_1\left(s^3+\chi_1 s^2+\delta_1 s+\lambda_1\right)}{\left(s^2+2\tau_{T_1}^{-1}s+2\tau_{T_1}^{-2}\right)}}\right)z_1,\quad \gamma_{21}(s)=sinh\left(\sqrt{\frac{\alpha_1\left(s^3+\chi_1 s^2+\delta_1 s+\lambda_1\right)}{\left(s^2+2\tau_{T_1}^{-1}s+2\tau_{T_1}^{-2}\right)}}\right)z_1$$

$$\gamma_{23}(s)=-A_2 cosh\left(\sqrt{\frac{\alpha_2\left(s^3+\chi_2 s^2+\delta_2 s+\lambda_2\right)}{\left(s^2+2\tau_{T_2}^{-1}s+2\tau_{T_2}^{-2}\right)}}\right)z_1,\quad \gamma_{24}(s)=-sinh\left(\sqrt{\frac{\alpha_2\left(s^3+\chi_2 s^2+\delta_2 s+\lambda_2\right)}{\left(s^2+2\tau_{T_2}^{-1}s+2\tau_{T_2}^{-2}\right)}}\right)z_1$$

$$\begin{aligned}\Omega_2(s)=&\frac{\mu_2}{\alpha_2\left(s^3+\chi_2 s^2+\delta_2 s+\lambda_2\right)}\left[\left(s^2+2\tau_{q_2}^{-1}s+2\tau_{q_2}^{-2}\right)+\xi_2\left(s+2^{-1}\tau_{q_2}\right)+\zeta_2\right]\tilde{T}_b\\&+\frac{\eta_2}{\alpha_2\left(s^3+\chi_2 s^2+\delta_2 s+\lambda_2\right)}\left[\begin{array}{l}\left(s^2+2\tau_{q_2}^{-1}s+2\tau_{q_2}^{-2}\right)\tilde{\tilde{q}}_m-\left(s+2^{-1}\tau_{q_2}\right)\tilde{q}_{b,0}\\\left(s^2+2\tau_{q_2}^{-1}s+2\tau_{q_2}^{-2}\right)\tilde{\tilde{q}}_{ext}-\left(s+2^{-1}\tau_{q_2}\right)\tilde{q}_{\text{ext},0}\end{array}\right]\\&-\frac{\mu_1}{\alpha_1\left(s^3+\chi_1 s^2+\delta_1 s+\lambda_1\right)}\left[\left(s^2+2\tau_{q_1}^{-1}s+2\tau_{q_1}^{-2}\right)+\xi_1\left(s+2^{-1}\tau_{q,1}\right)+\zeta_1\right]\tilde{T}_b\\&-\frac{\eta_1}{\alpha_1\left(s^3+\chi_1 s^2+\delta_1 s+\lambda_1\right)}\left[\begin{array}{l}\left(s^2+2\tau_{q_1}^{-1}s+2\tau_{q_1}^{-2}\right)\tilde{\tilde{q}}_m-\left(s+2^{-1}\tau_{q,1}\right)\tilde{q}_{b,0}\\\left(s^2+2\tau_{q_1}^{-1}s+2\tau_{q_1}^{-2}\right)\tilde{\tilde{q}}_{ext}-\left(s+2^{-1}\tau_{q,1}\right)\tilde{q}_{\text{ext},0}\end{array}\right]\end{aligned}$$

$$\gamma_{31}(s)=\left(\sqrt{\frac{\alpha_1\left(s^3+\chi_1 s^2+\delta_1 s+\lambda_1\right)}{\left(s^2+2\tau_{T_1}^{-1}s+2\tau_{T_1}^{-2}\right)}}\right)sinh\left(\sqrt{\frac{\alpha_1\left(s^3+\chi_1 s^2+\delta_1 s+\lambda_1\right)}{\left(s^2+2\tau_{T_1}^{-1}s+2\tau_{T_1}^{-2}\right)}}\right)z_1,$$

$$\gamma_{32}(s)=\left(\sqrt{\frac{\alpha_1\left(s^3+\chi_1 s^2+\delta_1 s+\lambda_1\right)}{\left(s^2+2\tau_{T_1}^{-1}s+2\tau_{T_1}^{-2}\right)}}\right)cosh\left(\sqrt{\frac{\alpha_1\left(s^3+\chi_1 s^2+\delta_1 s+\lambda_1\right)}{\left(s^2+2\tau_{T_1}^{-1}s+2\tau_{T_1}^{-2}\right)}}\right)z_1$$

$$\gamma_{33}(s)=-\left(\sqrt{\frac{\alpha_2\left(s^3+\chi_2 s^2+\delta_2 s+\lambda_2\right)}{\left(s^2+2\tau_{T_2}^{-1}s+2\tau_{T_2}^{-2}\right)}}\right)sinh\left(\sqrt{\frac{\alpha_2\left(s^3+\chi_2 s^2+\delta_2 s+\lambda_2\right)}{\left(s^2+2\tau_{T_2}^{-1}s+2\tau_{T_2}^{-2}\right)}}\right)z_1$$

$$\gamma_{34}(s) = -\left(\sqrt{\frac{\alpha_2\left(s^3+\chi_2 s^2+\delta_2 s+\lambda_2\right)}{\left(s^2+2\tau_{T_2}^{-1}s+2\tau_{T_2}^{-2}\right)}}\right) cosh\left(\sqrt{\frac{\alpha_2\left(s^3+\chi_2 s^2+\delta_2 s+\lambda_2\right)}{\left(s^2+2\tau_{T_2}^{-1}s+2\tau_{T_2}^{-2}\right)}}\right) z_1$$

$$\gamma_{43}(s) = cosh\left(\sqrt{\frac{\alpha_2\left(s^3+\chi_2 s^2+\delta_2 s+\lambda_2\right)}{\left(s^2+2\tau_{T_2}^{-1}s+2\tau_{T_2}^{-2}\right)}}\right) z_2, \quad \gamma_{44}(s) = sinh\left(\sqrt{\frac{\alpha_2\left(s^3+\chi_2 s^2+\delta_2 s+\lambda_2\right)}{\left(s^2+2\tau_{T_2}^{-1}s+2\tau_{T_2}^{-2}\right)}}\right) z_2$$

$$\gamma_{45}(s) = -A_3 cosh\left(\sqrt{\frac{\alpha_3\left(s^3+\chi_3 s^2+\delta_3 s+\lambda_3\right)}{\left(s^2+2\tau_{T_3}^{-1}s+2\tau_{T_3}^{-2}\right)}}\right) z_2, \quad \gamma_{46}(s) = -B_3 sinh\left(\sqrt{\frac{\alpha_3\left(s^3+\chi_3 s^2+\delta_3 s+\lambda_3\right)}{\left(s^2+2\tau_{T_3}^{-1}s+2\tau_{T_3}^{-2}\right)}}\right) z_2$$

$$\begin{aligned}
\Omega_4(s) = {} & \frac{\mu_3}{\alpha_3\left(s^3+\chi_3 s^2+\delta_3 s+\lambda_3\right)}\left[\left(s^2+2\tau_{q_3}^{-1}s+2\tau_{q_3}^{-2}\right)+\xi_3\left(s+2^{-1}\tau_{q_3}\right)+\zeta_3\right]\tilde{T}_b \\
& + \frac{\eta_2}{\alpha_3\left(s^3+\chi_3 s^2+\delta_3 s+\lambda_3\right)}\begin{bmatrix}\left(s^2+2\tau_{q_3}^{-1}s+2\tau_{q_3}^{-2}\right)\tilde{\tilde{q}}_m-\left(s+2^{-1}\tau_{q_3}\right)\tilde{q}_{b,0} \\ \left(s^2+2\tau_{q_3}^{-1}s+2\tau_{q_3}^{-2}\right)\tilde{\tilde{q}}_{ext}-\left(s+2^{-1}\tau_{q_3}\right)\tilde{q}_{\text{ext},0}\end{bmatrix} \\
& - \frac{\mu_2}{\alpha_2\left(s^3+\chi_2 s^2+\delta_2 s+\lambda_2\right)}\left[\left(s^2+2\tau_{q_2}^{-1}s+2\tau_{q_2}^{-2}\right)+\xi_2\left(s+2^{-1}\tau_{q_2}\right)+\zeta_2\right]\tilde{T}_b \\
& - \frac{\eta_2}{\alpha_2\left(s^3+\chi_2 s^2+\delta_2 s+\lambda_2\right)}\begin{bmatrix}\left(s^2+2\tau_{q_2}^{-1}s+2\tau_{q_2}^{-2}\right)\tilde{\tilde{q}}_m-\left(s+2^{-1}\tau_{q_2}\right)\tilde{q}_{b,0} \\ \left(s^2+2\tau_{q_2}^{-1}s+2\tau_{q_2}^{-2}\right)\tilde{\tilde{q}}_{ext}-\left(s+2^{-1}\tau_{q_2}\right)\tilde{q}_{\text{ext},0}\end{bmatrix}
\end{aligned}$$

$$\gamma_{53}(s) = \left(\sqrt{\frac{\alpha_2\left(s^3+\chi_2 s^2+\delta_2 s+\lambda_2\right)}{\left(s^2+2\tau_{T_2}^{-1}s+2\tau_{T_2}^{-2}\right)}}\right) sinh\left(\sqrt{\frac{\alpha_2\left(s^3+\chi_2 s^2+\delta_2 s+\lambda_2\right)}{\left(s^2+2\tau_{T_2}^{-1}s+2\tau_{T_2}^{-2}\right)}}\right) z_2,$$

$$\gamma_{54}(s) = \left(\sqrt{\frac{\alpha_2\left(s^3+\chi_2 s^2+\delta_2 s+\lambda_2\right)}{\left(s^2+2\tau_{T_2}^{-1}s+2\tau_{T_2}^{-2}\right)}}\right) cosh\left(\sqrt{\frac{\alpha_2\left(s^3+\chi_2 s^2+\delta_2 s+\lambda_2\right)}{\left(s^2+2\tau_{T_2}^{-1}s+2\tau_{T_2}^{-2}\right)}}\right) z_2,$$

$$\gamma_{55}(s) = -A_3\left(\sqrt{\frac{\alpha_3\left(s^3+\chi_3 s^2+\delta_3 s+\lambda_3\right)}{\left(s^2+2\tau_{T_3}^{-1}s+2\tau_{T_3}^{-2}\right)}}\right) sinh\left(\sqrt{\frac{\alpha_3\left(s^3+\chi_3 s^2+\delta_3 s+\lambda_3\right)}{\left(s^2+2\tau_{T_3}^{-1}s+2\tau_{T_3}^{-2}\right)}}\right) z_2,$$

$$\gamma_{56}(s) = -B_3\left(\sqrt{\frac{\alpha_3\left(s^3+\chi_3 s^2+\delta_3 s+\lambda_3\right)}{\left(s^2+2\tau_{T_3}^{-1}s+2\tau_{T_3}^{-2}\right)}}\right) cosh\left(\sqrt{\frac{\alpha_3\left(s^3+\chi_3 s^2+\delta_3 s+\lambda_3\right)}{\left(s^2+2\tau_{T_3}^{-1}s+2\tau_{T_3}^{-2}\right)}}\right) z_2$$

$$\gamma_{65}(s) = cosh\left(\sqrt{\frac{\alpha_3\left(s^3+\chi_3 s^2+\delta_3 s+\lambda_3\right)}{\left(s^2+2\tau_{T_3}^{-1}s+2\tau_{T_3}^{-2}\right)}}\right) H, \quad \gamma_{66}(s) = sinh\left(\sqrt{\frac{\alpha_3\left(s^3+\chi_3 s^2+\delta_3 s+\lambda_3\right)}{\left(s^2+2\tau_{T_3}^{-1}s+2\tau_{T_3}^{-2}\right)}}\right) H$$

$$\Omega_6(s) = \frac{\tilde{T}_b}{s} - \left\{\begin{aligned} & \frac{\mu_3}{\alpha_3\left(s^3+\chi_3 s^2+\delta_3 s+\lambda_3\right)}\left[\left(s^2+2\tau_{q_3}^{-1}s+2\tau_{q_3}^{-2}\right)+\xi_3\left(s+2^{-1}\tau_{q_3}\right)+\zeta_3\right]\tilde{T}_b \\ & + \frac{\eta_2}{\alpha_3\left(s^3+\chi_3 s^2+\delta_3 s+\lambda_3\right)}\begin{bmatrix}\left(s^2+2\tau_{q_3}^{-1}s+2\tau_{q_3}^{-2}\right)\tilde{\tilde{q}}_m-\left(s+2^{-1}\tau_{q_3}\right)\tilde{q}_{b,0} \\ \left(s^2+2\tau_{q_3}^{-1}s+2\tau_{q_3}^{-2}\right)\tilde{\tilde{q}}_{ext}-\left(s+2^{-1}\tau_{q_3}\right)\tilde{q}_{\text{ext},0}\end{bmatrix}\end{aligned}\right\}$$

Applying Crammer's rule to find $A_1$, $B_1$, $A_2$, $B_2$, $A_3$ *and* $B_3$, we have

$$A_1 = \frac{\begin{vmatrix} \Omega_1(s,\beta_{m1}) & \gamma_{12}(s) & 0 & 0 & 0 & 0 \\ \Omega_2(s) & \gamma_{22}(s) & \gamma_{23}(s) & \gamma_{24}(s) & 0 & 0 \\ 0 & \gamma_{32}(s) & \gamma_{33}(s) & \gamma_{34}(s) & 0 & 0 \\ \Omega_4(s) & 0 & \gamma_{43}(s) & \gamma_{44}(s) & \gamma_{45}(s) & \gamma_{46}(s) \\ 0 & 0 & \gamma_{53}(s) & \gamma_{54}(s) & \gamma_{55}(s) & \gamma_{56}(s) \\ 0 & 0 & 0 & 0 & \gamma_{65}(s) & \gamma_{66}(s) \end{vmatrix}}{\begin{vmatrix} 0 & \gamma_{12}(s) & 0 & 0 & 0 & 0 \\ \gamma_{21}(s) & \gamma_{22}(s) & \gamma_{23}(s) & \gamma_{24}(s) & 0 & 0 \\ \gamma_{31}(s) & \gamma_{32}(s) & \gamma_{33}(s) & \gamma_{34}(s) & 0 & 0 \\ 0 & 0 & \gamma_{43}(s) & \gamma_{44}(s) & \gamma_{45}(s) & \gamma_{46}(s) \\ 0 & 0 & \gamma_{53}(s) & \gamma_{54}(s) & \gamma_{55}(s) & \gamma_{56}(s) \\ 0 & 0 & 0 & 0 & \gamma_{65}(s) & \gamma_{66}(s) \end{vmatrix}} \tag{75}$$

$$B_1 = \frac{\begin{vmatrix} 0 & \Omega_1(s,\beta_{m1}) & 0 & 0 & 0 & 0 \\ \gamma_{21}(s) & \Omega_2(s) & \gamma_{23}(s) & \gamma_{24}(s) & 0 & 0 \\ \gamma_{31}(s) & 0 & \gamma_{33}(s) & \gamma_{34}(s) & 0 & 0 \\ 0 & \Omega_4(s) & \gamma_{43}(s) & \gamma_{44}(s) & \gamma_{45}(s) & \gamma_{46}(s) \\ 0 & 0 & \gamma_{53}(s) & \gamma_{54}(s) & \gamma_{55}(s) & \gamma_{56}(s) \\ 0 & 0 & 0 & 0 & \gamma_{65}(s) & \gamma_{66}(s) \end{vmatrix}}{\begin{vmatrix} 0 & \gamma_{12}(s) & 0 & 0 & 0 & 0 \\ \gamma_{21}(s) & \gamma_{22}(s) & \gamma_{23}(s) & \gamma_{24}(s) & 0 & 0 \\ \gamma_{31}(s) & \gamma_{32}(s) & \gamma_{33}(s) & \gamma_{34}(s) & 0 & 0 \\ 0 & 0 & \gamma_{43}(s) & \gamma_{44}(s) & \gamma_{45}(s) & \gamma_{46}(s) \\ 0 & 0 & \gamma_{53}(s) & \gamma_{54}(s) & \gamma_{55}(s) & \gamma_{56}(s) \\ 0 & 0 & 0 & 0 & \gamma_{65}(s) & \gamma_{66}(s) \end{vmatrix}} \tag{76}$$

$$A_2 = \frac{\begin{vmatrix} 0 & \gamma_{12}(s) & \Omega_1(s,\beta_{m1}) & 0 & 0 & 0 \\ \gamma_{21}(s) & \gamma_{22}(s) & \Omega_2(s) & \gamma_{24}(s) & 0 & 0 \\ \gamma_{31}(s) & \gamma_{32}(s) & 0 & \gamma_{34}(s) & 0 & 0 \\ 0 & 0 & \Omega_4(s) & \gamma_{44}(s) & \gamma_{45}(s) & \gamma_{46}(s) \\ 0 & 0 & 0 & \gamma_{54}(s) & \gamma_{55}(s) & \gamma_{56}(s) \\ 0 & 0 & 0 & 0 & \gamma_{65}(s) & \gamma_{66}(s) \end{vmatrix}}{\begin{vmatrix} 0 & \gamma_{12}(s) & 0 & 0 & 0 & 0 \\ \gamma_{21}(s) & \gamma_{22}(s) & \gamma_{23}(s) & \gamma_{24}(s) & 0 & 0 \\ \gamma_{31}(s) & \gamma_{32}(s) & \gamma_{33}(s) & \gamma_{34}(s) & 0 & 0 \\ 0 & 0 & \gamma_{43}(s) & \gamma_{44}(s) & \gamma_{45}(s) & \gamma_{46}(s) \\ 0 & 0 & \gamma_{53}(s) & \gamma_{54}(s) & \gamma_{55}(s) & \gamma_{56}(s) \\ 0 & 0 & 0 & 0 & \gamma_{65}(s) & \gamma_{66}(s) \end{vmatrix}} \tag{77}$$

$$B_2 = \frac{\begin{vmatrix} 0 & \gamma_{12}(s) & 0 & \Omega_1(s,\beta_{m1}) & 0 & 0 \\ \gamma_{21}(s) & \gamma_{22}(s) & \gamma_{23}(s) & \Omega_2(s) & 0 & 0 \\ \gamma_{31}(s) & \gamma_{32}(s) & \gamma_{33}(s) & 0 & 0 & 0 \\ 0 & 0 & \gamma_{43}(s) & \Omega_4(s) & \gamma_{45}(s) & \gamma_{46}(s) \\ 0 & 0 & \gamma_{53}(s) & 0 & \gamma_{55}(s) & \gamma_{56}(s) \\ 0 & 0 & 0 & 0 & \gamma_{65}(s) & \gamma_{66}(s) \end{vmatrix}}{\begin{vmatrix} 0 & \gamma_{12}(s) & 0 & 0 & 0 & 0 \\ \gamma_{21}(s) & \gamma_{22}(s) & \gamma_{23}(s) & \gamma_{24}(s) & 0 & 0 \\ \gamma_{31}(s) & \gamma_{32}(s) & \gamma_{33}(s) & \gamma_{34}(s) & 0 & 0 \\ 0 & 0 & \gamma_{43}(s) & \gamma_{44}(s) & \gamma_{45}(s) & \gamma_{46}(s) \\ 0 & 0 & \gamma_{53}(s) & \gamma_{54}(s) & \gamma_{55}(s) & \gamma_{56}(s) \\ 0 & 0 & 0 & 0 & \gamma_{65}(s) & \gamma_{66}(s) \end{vmatrix}} \tag{78}$$

$$A_3 = \frac{\begin{vmatrix} 0 & \gamma_{12}(s) & 0 & 0 & \Omega_1(s,\beta_{m1}) & 0 \\ \gamma_{21}(s) & \gamma_{22}(s) & \gamma_{23}(s) & \gamma_{24}(s) & \Omega_2(s) & 0 \\ \gamma_{31}(s) & \gamma_{32}(s) & \gamma_{33}(s) & \gamma_{34}(s) & 0 & 0 \\ 0 & 0 & \gamma_{43}(s) & \gamma_{44}(s) & \Omega_4(s) & \gamma_{46}(s) \\ 0 & 0 & \gamma_{53}(s) & \gamma_{54}(s) & 0 & \gamma_{56}(s) \\ 0 & 0 & 0 & 0 & 0 & \gamma_{66}(s) \end{vmatrix}}{\begin{vmatrix} 0 & \gamma_{12}(s) & 0 & 0 & 0 & 0 \\ \gamma_{21}(s) & \gamma_{22}(s) & \gamma_{23}(s) & \gamma_{24}(s) & 0 & 0 \\ \gamma_{31}(s) & \gamma_{32}(s) & \gamma_{33}(s) & \gamma_{34}(s) & 0 & 0 \\ 0 & 0 & \gamma_{43}(s) & \gamma_{44}(s) & \gamma_{45}(s) & \gamma_{46}(s) \\ 0 & 0 & \gamma_{53}(s) & \gamma_{54}(s) & \gamma_{55}(s) & \gamma_{56} \\ 0 & 0 & 0 & 0 & \gamma_{65}(s) & \gamma_{66}(s) \end{vmatrix}} \tag{79}$$

$$B_3 = \frac{\begin{vmatrix} 0 & \gamma_{12}(s) & 0 & 0 & 0 & \Omega_1(s,\beta_{m1}) \\ \gamma_{21}(s) & \gamma_{22}(s) & \gamma_{23}(s) & \gamma_{24}(s) & 0 & \Omega_2(s) \\ \gamma_{31}(s) & \gamma_{32}(s) & \gamma_{33}(s) & \gamma_{34}(s) & 0 & 0 \\ 0 & 0 & \gamma_{43}(s) & \gamma_{44}(s) & \gamma_{45}(s) & \Omega_4(s) \\ 0 & 0 & \gamma_{53}(s) & \gamma_{54}(s) & \gamma_{55}(s) & 0 \\ 0 & 0 & 0 & 0 & \gamma_{65}(s) & 0 \end{vmatrix}}{\begin{vmatrix} 0 & \gamma_{12}(s) & 0 & 0 & 0 & 0 \\ \gamma_{21}(s) & \gamma_{22}(s) & \gamma_{23}(s) & \gamma_{24}(s) & 0 & 0 \\ \gamma_{31}(s) & \gamma_{32}(s) & \gamma_{33}(s) & \gamma_{34}(s) & 0 & 0 \\ 0 & 0 & \gamma_{43}(s) & \gamma_{44}(s) & \gamma_{45}(s) & \gamma_{46}(s) \\ 0 & 0 & \gamma_{53}(s) & \gamma_{54}(s) & \gamma_{55}(s) & \gamma_{56}(s) \\ 0 & 0 & 0 & 0 & \gamma_{65}(s) & \gamma_{66}(s) \end{vmatrix}} \tag{80}$$

The solution in eqs. (75) – (80) is substituted in Eqs. (70) – (72) and then the inverse Fourier transforms are found as:

$$\bar{T}_1(\beta_{m1},z,s)=\frac{2}{L}\sum_{m=1}^{\infty}\left\{\begin{array}{l}A_1(\beta_{m1},z_1,s)cosh\left(\sqrt{\frac{\alpha_1\left(s^3+\chi_1 s^2+\delta_1 s+\lambda_1\right)}{\left(s^2+2\tau_{T_1}^{-1}s+2\tau_{T_1}^{-2}\right)}}\right)z\\+B_1(\beta_{m1},z_1,s)sinh\left(\sqrt{\frac{\alpha_1\left(s^3+\chi_1 s^2+\delta_1 s+\lambda_1\right)}{\left(s^2+2\tau_{T_1}^{-1}s+2\tau_{T_1}^{-2}\right)}}\right)z\\+\frac{\mu_1}{\alpha_1\left(s^3+\chi_1 s^2+\delta_1 s+\lambda_1\right)}\left[\left(s^2+2\tau_{q_1}^{-1}s+2\tau_{q_1}^{-2}\right)+\xi_1\left(s+2^{-1}\tau_{q,1}\right)+\zeta_1\right]\tilde{T}_b\\+\frac{\eta_1}{\alpha_1\left(s^3+\chi_1 s^2+\delta_1 s+\lambda_1\right)}\left[\begin{array}{l}\left(s^2+2\tau_{q_1}^{-1}s+2\tau_{q_1}^{-2}\right)\tilde{\bar{q}}_m-\left(s+2^{-1}\tau_{q,1}\right)\tilde{q}_{b,0}\\\left(s^2+2\tau_{q_1}^{-1}s+2\tau_{q_1}^{-2}\right)\tilde{\bar{q}}_{ext}-\left(s+2^{-1}\tau_{q,1}\right)\tilde{q}_{\text{ext},0}\end{array}\right]\end{array}\right\}cos\left(\beta_{m1}x\right) \tag{81}$$

$$\bar{T}_2(\beta_{m2},z,s)=\frac{2}{L}\sum_{m=1}^{\infty}\left\{\begin{array}{l}A_2(\beta_{m2},z_1,z_2,s)cosh\left(\sqrt{\frac{\alpha_2\left(s^3+\chi_2 s^2+\delta_2 s+\lambda_2\right)}{\left(s^2+2\tau_{T_2}^{-1}s+2\tau_{T_2}^{-2}\right)}}\right)z\\+B_2(\beta_{m2},z_1,z_2,s)sinh\left(\sqrt{\frac{\alpha_2\left(s^3+\chi_2 s^2+\delta_2 s+\lambda_2\right)}{\left(s^2+2\tau_{T_2}^{-1}s+2\tau_{T_2}^{-2}\right)}}\right)z\\+\frac{\mu_2}{\alpha_2\left(s^3+\chi_2 s^2+\delta_2 s+\lambda_2\right)}\left[\left(s^2+2\tau_{q_2}^{-1}s+2\tau_{q_2}^{-2}\right)+\xi_2\left(s+2^{-1}\tau_{q_2}\right)+\zeta_2\right]\tilde{T}_b\\+\frac{\eta_2}{\alpha_2\left(s^3+\chi_2 s^2+\delta_2 s+\lambda_2\right)}\left[\begin{array}{l}\left(s^2+2\tau_{q_2}^{-1}s+2\tau_{q_2}^{-2}\right)\tilde{\bar{q}}_m-\left(s+2^{-1}\tau_{q_2}\right)\tilde{q}_{b,0}\\\left(s^2+2\tau_{q_2}^{-1}s+2\tau_{q_2}^{-2}\right)\tilde{\bar{q}}_{ext}-\left(s+2^{-1}\tau_{q_2}\right)\tilde{q}_{\text{ext},0}\end{array}\right]\end{array}\right\}cos\left(\beta_{m2}x\right) \tag{82}$$

$$\bar{T}_3(\beta_{m3},z,s)=\frac{2}{L}\sum_{m=1}^{\infty}\left\{\begin{array}{l}A_3(\beta_{m3},z_3,s)cosh\left(\sqrt{\frac{\alpha_3\left(s^3+\chi_3 s^2+\delta_3 s+\lambda_3\right)}{\left(s^2+2\tau_{T_3}^{-1}s+2\tau_{T_3}^{-2}\right)}}\right)z\\+B_3(\beta_{m3},z_3,s)\left(sinh\sqrt{\frac{\alpha_3\left(s^3+\chi_3 s^2+\delta_3 s+\lambda_3\right)}{\left(s^2+2\tau_{T_3}^{-1}s+2\tau_{T_3}^{-2}\right)}}\right)z\\+\frac{\mu_3}{\alpha_3\left(s^3+\chi_3 s^2+\delta_3 s+\lambda_3\right)}\left[\left(s^2+2\tau_{q_3}^{-1}s+2\tau_{q_3}^{-2}\right)+\xi_3\left(s+2^{-1}\tau_{q_3}\right)+\zeta_3\right]\tilde{T}_b\\+\frac{\eta_2}{\alpha_3\left(s^3+\chi_3 s^2+\delta_3 s+\lambda_3\right)}\left[\begin{array}{l}\left(s^2+2\tau_{q_3}^{-1}s+2\tau_{q_3}^{-2}\right)\tilde{\bar{q}}_m-\left(s+2^{-1}\tau_{q_3}\right)\tilde{q}_{b,0}\\\left(s^2+2\tau_{q_3}^{-1}s+2\tau_{q_3}^{-2}\right)\tilde{\bar{q}}_{ext}-\left(s+2^{-1}\tau_{q_3}\right)\tilde{q}_{\text{ext},0}\end{array}\right]\end{array}\right\}cos\left(\beta_{m3}x\right) \tag{83}$$

Since the analytical inversion of the included Laplace transforms in the solutions are very complex. The developed solutions are numerically evaluated using Simon et al. [36] given as

$$T_1\left(x,z,t\right)=\frac{e^{a_p t}}{t}\left[\frac{1}{2}\bar{T}_1\left(x,z,a_p\right)+Re\sum_{n=1}^{N}\left[\bar{T}_1\left(x,z,a_p+i\frac{n\pi}{t}\right)\right]\left(-1\right)^n\right] \tag{84}$$

$$T_2\left(x,z,t\right)=\frac{e^{a_p t}}{t}\left[\frac{1}{2}\bar{T}_2\left(x,z,a_p\right)+Re\sum_{n=1}^{N}\left[\bar{T}_2\left(x,z,a_p+i\frac{n\pi}{t}\right)\right]\left(-1\right)^n\right] \tag{85}$$

$$T_3(x,z,t)=\frac{e^{a_p t}}{t}\left[\frac{1}{2}\bar{T}_3(x,z,a_p)+Re\sum_{n=1}^{N}\left[\bar{T}_3\left(x,z,a_p+i\frac{n\pi}{t}\right)\right](-1)^n\right] \tag{86}$$

As stated before, the optimal value is given as

$$a_v t=4.7 \tag{87}$$

**6. Thermal Damage Model and Classification of Degree of Burns**

Skin thermal damage starts when the basal epidermis temperature reaches 44°C. To evaluate the quantity and the rate of thermal damage to the skin tissues, the following rate model is developed.

$$\frac{d\Omega_l}{dt}=\begin{cases}0 & T<44^oC\\ Ae^{-\frac{E_a}{RT_l(x,z,t)}} & T\geq 44^oC\end{cases} \tag{88}$$

where $\Omega, t, A, E_a, R$ *and* $T_l$ represents the thermal damage in tissue, time, frequency factor, the activation energy for skin, universal gas constant and temperature of layer, *l*.

On integrating the above, we have

$$\Omega_l=\begin{cases}0 & T<44^oC\\ \int_0^t Ae^{-\frac{E_a}{RT_l(x,z,t)}}dt & T\geq 44^oC\end{cases} \tag{89}$$

The above skin burn model predicts the thermal damage, which is the most important parameter to be predicted to determine the degree of burn for clinical decision to be taken for therapeutic treatment of the burnt skin [37]. The value of the thermal damage $\Omega$, is used to classify the burn injuries as first-degree, second-degree and third-degree according to their depth as presented in Table 1.

Table 1. Burn degree and its associated biological features

| **Burn** | $\Omega$ | **Biological features** |
|---|---|---|
| First-degree (Superficial or epidermal burns) | 0.53 | affects epidermis with vasodilatation of the sub-capillary vessels, redness of affected area with no permanent scars or discolouration, mild pain and healing is rapid |
| Second-degree (Partial-thickness burns) | 1.0 | Both epidermis and dermis are slightly affected. Burn can be superficial or deep. Superficial burn results in moist blisters, whilst deep burn affects the capillaries or blood vessels causing tissue edema and blisters on the skin |
| Third-degree (Full Thickness) | 10000 | Both epidermis and dermis are thermally damaged, causing blood flow to stop. The cells around the burn region start to die leading to leathery skin. Recovery from burn degree requires special treatment |

It is worth noting that based on a thorough understanding of the burn degree classification the developed skin burns model is used to determine the required time to generate the different degree of burns on the human body.

## 7. Discussion

The above-developed solutions are simulated and the effects of various thermal and flow parameters on the temperature and the assessment of burn injury are investigated. The simulations of the burn injury are carried at the epidermis–dermis interface at a depth of 7.5mm from the skin surface and 2.0 mm from the symmetry line i.e. (x, y) = (7.5 mm, 2.0 mm). The effects of thermal conductivity of the triple layer of the skin for prediction of burn injury threshold are presented in Figs. 3, 4 and 5. In Fig. 3, as the thermal conductivity of the epidermis increases by a factor of 2, the injury threshold shifts towards the left-hand side of the base value (0.210 W/mK). This is because, as the thermal conductivity of the epidermis increases under continuous temperature exposure, the thermal resistance reduces resulting in increased heat penetration of the tissue. Under this condition, the time required to reach a second-degree burn injury situation becomes minimal, which implies that the higher the thermal conductivity of the skin layer tissue, the lower the degree of burn injury.

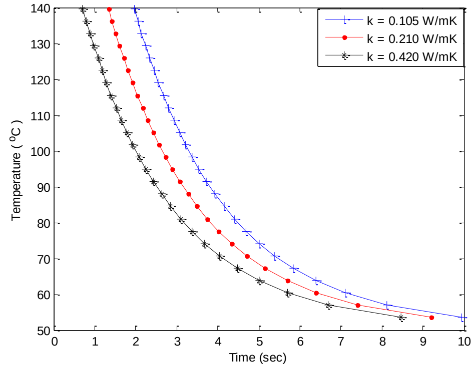


Fig. 3. Effects of thermal conductivity of the epidermis on the prediction of burn injury threshold

Conversely, an opposite trend is observed for decrease thermal conductivity by a factor of 2, resulting in the injury threshold to shift towards the right-hand side of the base value (0.210 W/mK). This is because as thermal conductivity reduces, the thermal resistance increases resulting in reduced heat penetration of the tissue. Consequently, under such burn condition, more time would be required to reach a second-degree burn injury.

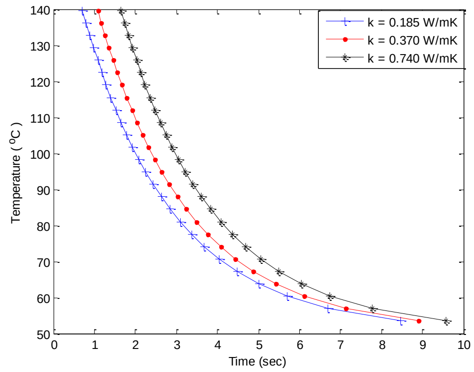


Fig. 4. Effects of dermis thermal conductivity on the prediction of burn injury threshold

In Fig. 4, it is shown that as the thermal conductivity of the dermis increases by a factor of 2, the injury threshold shifts towards the right-hand side of the base value (i.e. 0.370 W/mK). This phenomenon occurs, as an increase in the dermis thermal conductivity causes heat at the epidermis-dermis interface to be readily transferred to the deeper tissue and consequently resulting in more time required to reach the injury threshold. However, an opposite trend is observed when the thermal conductivity of the dermis is decreased by a factor of 2, resulting in the injury threshold to shift towards the left-hand side of the base value (0.370 W/mK). Moreover, the above finding implies that the thermal conductivity of the dermis reduces as the temperature of exposure heat decreases. From Fig. 5, the thermal conductivity of the hypodermis exhibits no significant effect on the prediction of the injury threshold.

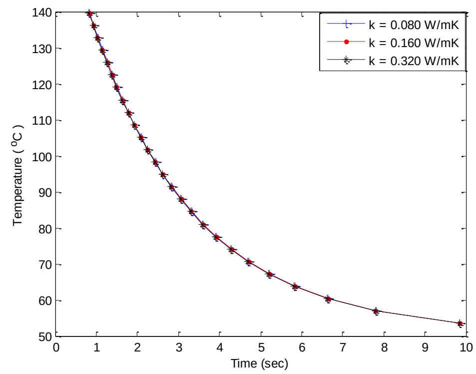


Fig. 5. Effects of hypodermis thermal conductivity on the prediction of burn injury threshold

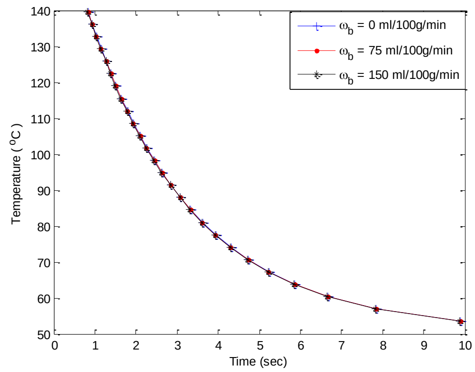


Fig. 6. Effects of blood perfusion rate on the prediction of burn injury threshold

Fig. 6 shows the effect of blood perfusion rate on the prediction threshold. Under constant volumetric capacity of the blood, the effects of no perfusion (0ml/100g/min), half of the maximum dilatation (75ml/100g/min) and maximum blood flow (150ml/100g/min) of the skin vessels. From Fig. 6, it is shown that the blood perfusion rate exhibits no significant effect on the dermis and invariably does not affect the burn injury prediction in the dermis layer. The effects of initial skin tissue temperature on the burn injury threshold prediction are presented in Fig. 7. From Fig. 7, the initial skin tissue temperature exhibits significant effects on skin exposed to burn injury. This is because the warmth of the skin tissue increases as the temperature experienced throughout the heating and cooling periods increases resulting in an increased degree of burn injury.

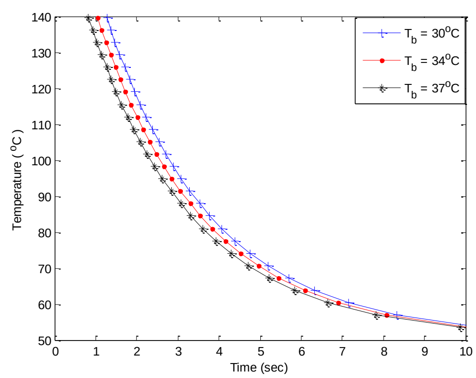


Fig. 7. Effects of initial skin tissue temperature on the prediction of burn injury threshold

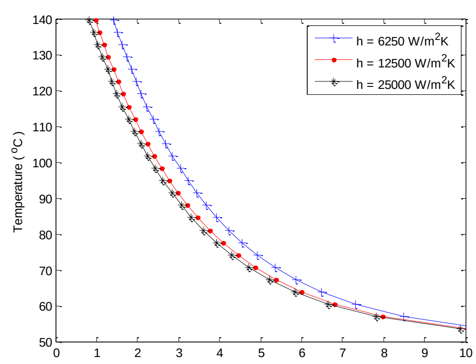


Fig. 8. Effects of convective heat transfer coefficient on the prediction of burn injury threshold

The effect of convective heat transfer coefficient for prediction of burn injury threshold is illustrated in Fig. 8. Fig. 8 shows that the heat transfer coefficient exhibits significant effects on the burn injury threshold for minimal exposure time. However, the effect of the convective heat transfer coefficient becomes negligible, with prolonged exposure time. This is because, under prolonged exposure of the skin layers to burn, the burn injury is dominated by conductive heat transfer in the tissue rather than the convective heat transfer at the surface of the skin. Moreover, as the convective heat transfer coefficient increases by a factor of 2, the injury threshold shifts towards the left-hand side of the base value (1250 W/m$^2$K). Under this condition, minimal time is needed for the burn to reach the second-degree injury threshold. However, when the convective heat transfer coefficient is decreased by a factor of 2, the injury threshold shifts towards the right-hand side of the base value (1250 W/m$^2$K), which shows that as convective heat transfer coefficient decreases, more time is needed for the skin burn to reach a second-degree burn injury.

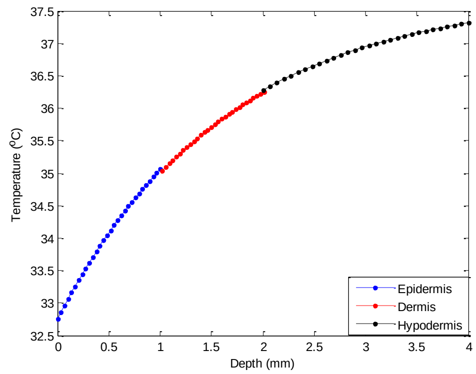


Fig. 9. Initial tissue temperature profiles for three-layer skin tissue

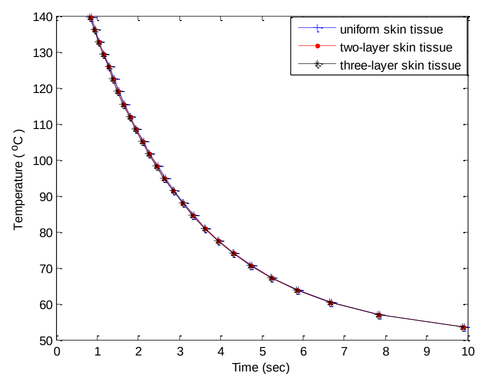


Fig. 10. Injury threshold using a uniform tissue temperature of 34°C and initial temperature profile for two-layer and three-layer slab

Fig. 9 shows the initial tissue temperature profiles for the three-layer skin tissue whilst the injury thresholds using a uniform tissue temperature of 34°C and the initial temperature profile of the two-layer and three-layer slab is presented in Fig. 10. From Figs 9 and 10, the selection of initial temperature of 34°C for all the three layers is highly reasonable. This implies that for accurate analysis of skin tissue burn injury, it is convenient to use a uniform initial temperature for all three layers. In addition, the skin injury layer threshold range for different values of thermal properties is presented in Fig. 11. In Fig. 11, the effect of a worst-case combination of parameters is considered where the upper limit represents a combination of low thermal properties of skin components, low heat transfer coefficient and low initial tissue temperature. Conversely, the lower limit represents a combination of high values for these parameters.

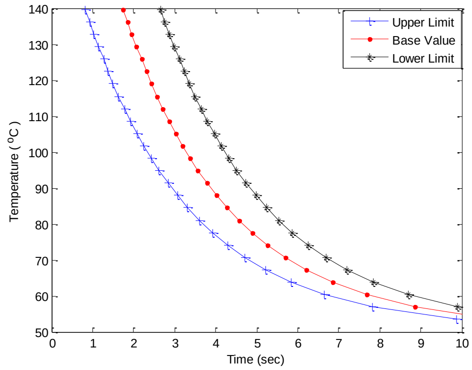

Fig. 11. Skin injury layer threshold range of combinations of different values of thermal properties

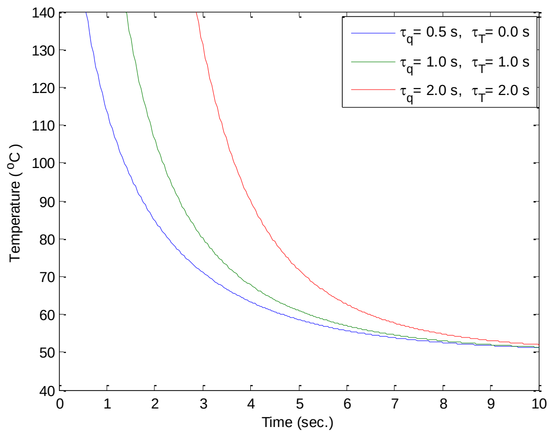

Fig. 12. Effects of relaxation time and thermalisation time on the skin temperature

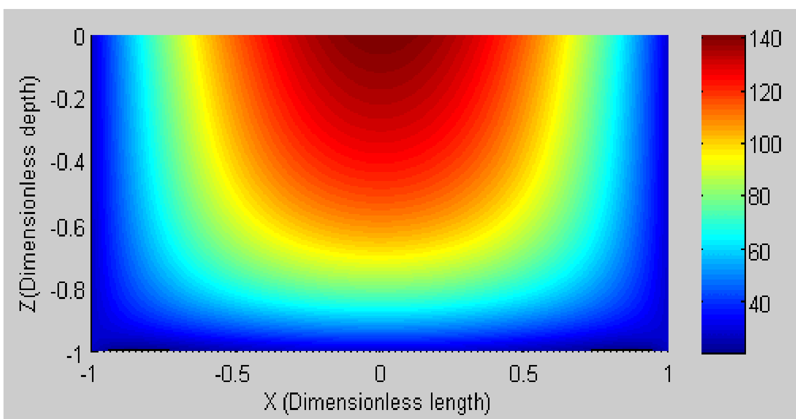

Fig. 13. Temperature distribution of the skin tissue

The effects of relaxation time and thermalisation time on the skin temperature during the cooling process is presented in Fig. 12. The simulation shows that the peak temperature predicted by the classical bioheat equation is higher than the DPL during the heating process. However, the temperatures from all considered cases converge during the cooling process. This is because the finite speed of wave propagation and the peak temperature predicted by the DPL occurs with time lags. The time lag causes longer thermal dissipation, i.e., cooling by the heat conduction of tissue and by the blood perfusion period of peak temperature. The predicted higher temperature by the classical bioheat model than the DPL model implies that the accumulation of the thermal damage is overestimated since the level of the accumulative thermal damage depends primarily on the peak temperature. Moreover, as seen in Fig. 12, the relaxation time and thermalisation time enhances the heat diffusion and reduces the thermal damage in the skin tissues. This effect is noticeable as the heat transfer process approaches the peak temperature during heating. However, the curves are close and almost similar values are predicted during cooling for the relaxation time and thermalisation time considered. The temperature distribution in the skin tissue at depth 0.006m and length 0.024m for a period of 8sec is presented in Fig. 13. Fig. 13 is presented of the dimensionless length and depth and the negative values in the dimensionless depth indicate depth zero i.e., below the skin surface. The dimensional length ranges from –1 to +1 to depict symmetry as stated in the boundary conditions. Further, the results of the present study are compared with the different established works in the literature including Henriques [38], Fugitt et al. [39], Stoll and Greene [40], Takata [41] and Wu [42] as presented in Figs. 14-19. Fig. 14 demonstrates the comparison of the various results for the effects of surface temperature on burn injury. Also, Figs. 15 - 17 demonstrates the required time to reach the first, second and third-degree burn injuries when the skin tissue temperature is maintained at 50°C, 70°C, and 90°C respectively. The effects of tissue depth and surface temperature on the degree of burn and burn injury are illustrated in Figs. 18 and 19. It is shown that the thermal damage value or the value of the burn injury is minimal for low tissue temperature but increases rapidly with temperature above 50°C.

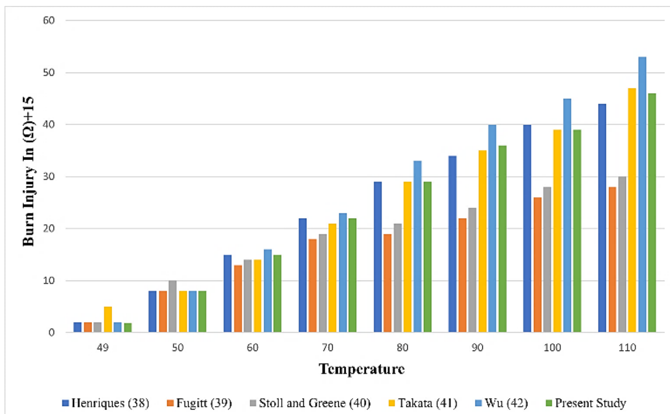


Fig. 14. Comparison of present work with existing studies on the effect of surface temperature on burn injury

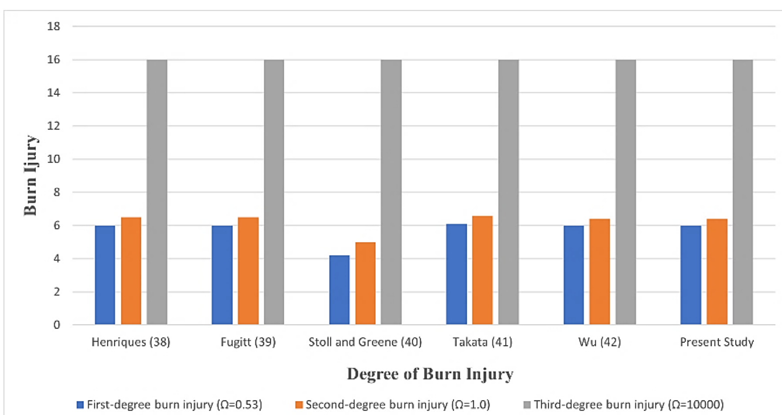


Fig. 15. Comparison of present work with existing studies on the degree of burn and burn injury time (log) at a fixed surface temperature of 50ºC

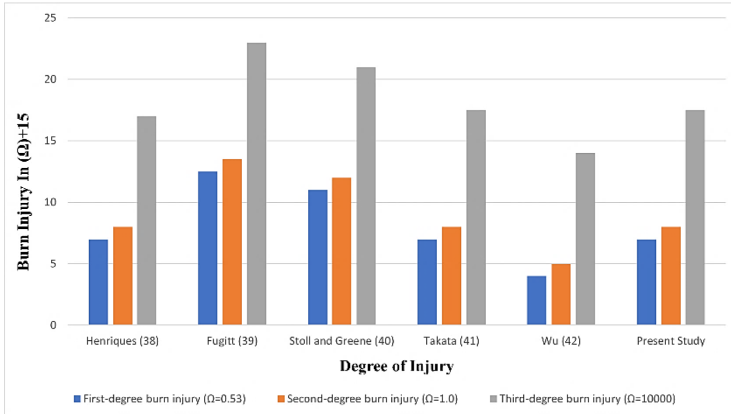


Fig. 16. Comparison of present work with existing studies on the degree of burn and burn injury time (log) at a fixed surface temperature of 70ºC

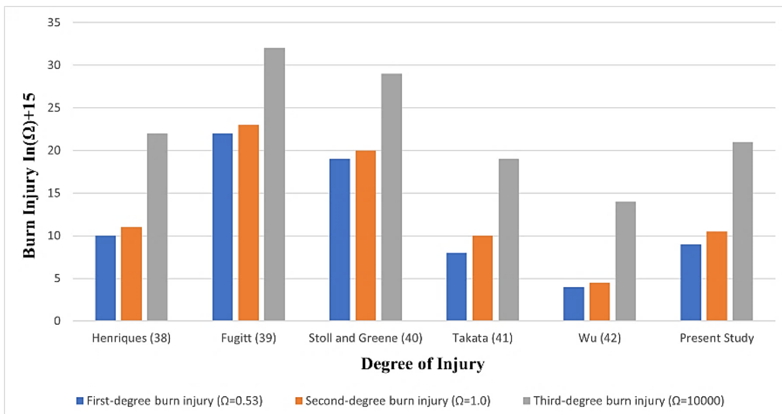


Fig. 17. Comparison of present work with existing studies on the degree of burn and burn injury time (log) at a fixed surface temperature of 90ºC

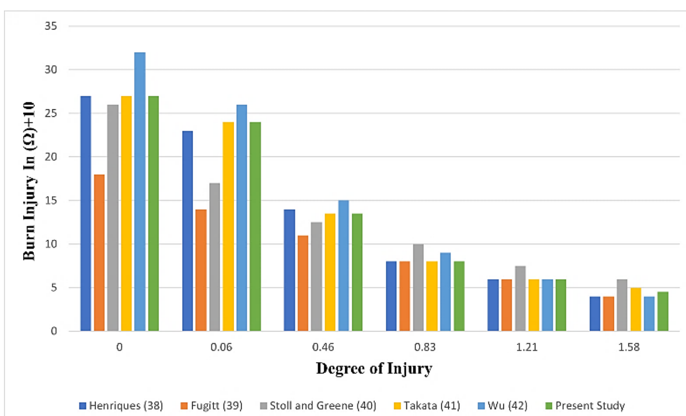


Fig. 18. Comparison of present work with existing studies on the degree of burn injury for various tissue depth at 80ºC for 5s

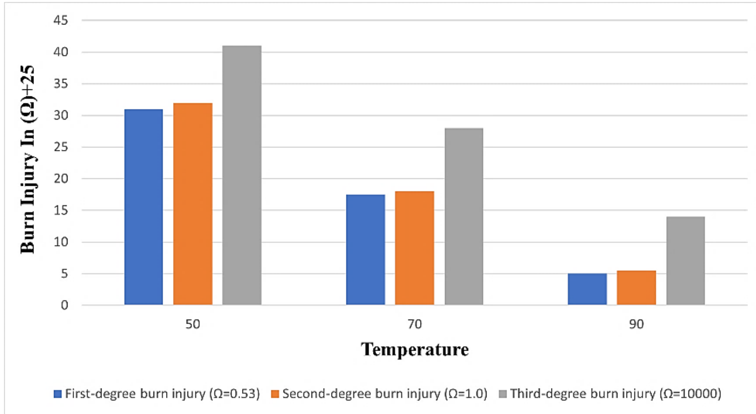


Fig. 19. Comparison of results for the effects of temperature on the degree of burn and its associated level of injury

Figs. 14-19 show that the present study quantitatively agrees with previous studies for low-level damage at temperatures below 50ºC. However, there is significant variation among the skin burn damage models results for high exposure temperatures. In fact, for temperature above 50ºC, only Henriques [38] and Takata [41] corresponds reasonably for all temperatures evaluated and in agreement with the results of the present study. Fig. 20 presents the validation of the present study with the experimental work of Takata [41]. From the comparison, the present study shows significant agreements with the experimental work of Takata [41], which demonstrates the reliability of the obtained results of the study.

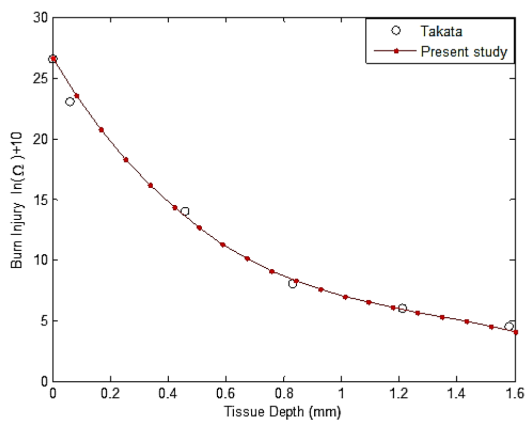


Fig. 20. Comparison of the present study on burn injury with established experimental results

## 8. Conclusion

In the present work, a non-Fourier prediction methodology of triple-layer human skin tissue for determining skin burn injury with non-ideal properties of tissue, metabolism, and blood perfusion. The DPL bioheat model of the triple-layer cutaneous tissue is solved analytically using Laplace and Fourier transform methods. From the detailed study, the following concluding remarks are drawn:

- An increase in the thermal conductivity of the epidermis decreases the thermal resistance, which readily causes increased heat penetration of the tissue, which implies that the higher the thermal conductivity of the tissue, the lower the degree of burn injury
- The study shows that an increase in dermis thermal conductivity results in a prolonged time to reach the injury threshold and the blood perfusion rate exhibits no net effect on the prediction of burn injury in the dermis layer. However, the initial skin tissue temperature exhibit significant effects on burn injury exposure

- The relaxation time and thermalisation time plays fundamental roles in the analysis of burn injury. The relaxation time and thermalisation time enhances the heat diffusion and reduces the thermal damage in the skin tissues
- The heat transfer coefficient plays significant effects on burn injury for small exposure time. The effect of the convective heat transfer coefficient becomes minimal for prolonged exposure time.
- For accurate analysis of the skin tissue burn injury, it is convenient to use uniform initial temperature for the triple-layer cutaneous layers
- The most dominating factors in the burn injury follow the order: relaxation time and thermalisation time, initial tissue temperature followed by the epidermis layer thermal conductivity, dermis layer thermal conductivity and convective heat transfer coefficient

The present work would help in the quantification of skin burn, and for clinicians and biomedical engineers to experiment, design, characterise and optimise strategies for delivering thermal therapies.

**Nomenclature**

| | |
|---|---|
| $c$ | Specific heat of tissue, $JKg^{-1}K^{-1}$ |
| $c_b$ | Specific heat of blood, $JKg^{-1}K^{-1}$ |
| $C$ | Thermal wave speed, $ms^{-1}$ |
| $E_a$ | Activation energy of denaturation reaction, $Jmol^{-1}$ |
| $k$ | Tissue thermal conductivity, $Wm^{-1}K^{-1}$ |
| $l$ | Bromwich contour integration line |
| $L$ | Tissue slab length, $m$ |
| $P$ | Frequency factor, $s^{-1}$ |
| $q$ | Heat flux density, $Wm^{-2}$ |
| $Q_m$ | Metabolic heat generation, $Wm^{-3}$ |
| $R$ | Universal gas constant, $Jmol^{-1}K^{-1}$ |
| $s$ | Laplace domain parameter |
| $t$ | Time variable, $s$ |
| $T$ | Tissue temperature, $\left(^{\circ}C\right)$ |
| $T_0$ | Initial tissue temperature, $\left(^{\circ}C\right)$ |
| $T_b$ | Blood temperature, $\left(^{\circ}C\right)$ |
| $\omega_b$ | Blood perfusion rate, $s^{-1}$ |
| $x, z$ | Coordinate variable, $m$ |

**Greek symbols**

| | |
|---|---|
| $\Omega$ | Thermal damage parameter |
| $\tau_T$ | Delay times for phase lag of microstructural interaction |
| $\tau_q$ | Delay times for phase lag of the heat flux |

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