Birmingham City
Business School

BIRMINGHAM CITY
University

# CAFÉ WORKING PAPER NO.9

## Centre for Applied Finance and Economics (CAFÉ)

## R&D-based Economic Growth in a Supermultiplier Model

Önder Nomaler, Danilo Spinola and Bart Verspagen

January 2021

# R&D-based Economic Growth in a Supermultiplier Model

Önder Nomaler[*], Danilo Spinola[†] & Bart Verspagen[‡]

**Abstract:**

We investigate how economic growth in a demand-led economy with semi-endogenous productivity growth can be compatible with a stable employment path. Our model uses a Sraffian supermultiplier (SSM), and we endogenize the growth rate of autonomous demand, and semi-endogenize productivity growth. The basic model has a steady state that is consistent with a stable employment rate. Consumption smoothing (between periods of high and low employment) by workers is the mechanism that keeps the growing economy stable. We also introduce a version of the model where the burden for stabilization falls upon government fiscal policy. This also yields a stable growth path, although the parameter restrictions for stability are more demanding in this case.

**Keywords:** Economic growth model, Sraffian supermultiplier, Research and Development (R&D)

**JEL Codes:** O41, E11, E12, E62

[*] UNU-Merit (United Nations University - Maastricht Economic and Social Research Institute on Innovation and Technology) & Maastricht University. nomaler@merit.unu.edu

[†] Birmingham City Business School & UNU-Merit. danilo.spinola@bcu.ac.uk

[‡] UNU-Merit & Maastricht University. verspagen@merit.unu.edu

## 1. Introduction

The core of Keynesian economics is the rejection of Say's Law, or, phrased in a more positive way, the importance of the demand side of the economy. In macroeconomics, this insight is applied, among other things, to the problem of unemployment (e.g., Mitchell et al., 2019). This is usually done in the context of the business cycle, i.e., without consideration of the long-run growth potential of the economy as expressed, for example, by productivity growth. However, Keynesian approaches to growth also exist, e.g., Kaldor (1957), Pasinetti (1981), Freitas and Serrano (2015) to name only a few (see Blecker and Setterfield 2020 for an extensive overview).

Productivity growth (especially labour productivity growth) is also an important (supply-side) source of economic growth (Maddison, 1991). In turn technological change is the main source of long-run productivity growth, and the literature on economic growth has therefore identified investment in technological change as a prime driver of growth. This puts Research and Development (R&D) at the center of analysis (e.g., Aghion and Howitt, 1992 in the mainstream tradition, or Nelson & Winter, 1982 and Silverberg and Verspagen, 1994 in the evolutionary tradition). However, most of these approaches assume that demand automatically adjusts to supply, and hence that growth is primarily a supply-side process.

In this paper, we present and analyze a model in which we have R&D based productivity growth, but also explicitly model the way in which demand adjusts in the long run. The key question that this poses is whether demand adjustment and productivity growth will simultaneously yield a stable long-run employment path. If demand grows persistently slower than productivity, the economy will tend towards a zero employment rate, whereas if demand grows persistently faster than productivity, labour will become a bottleneck for growth. In our model, demand and productivity growth are seen to adjust to each other, and a dynamic macroeconomic steady state emerges in which demand is the main economic coordination mechanism rather than price flexibility (Meijers et al., 2019).

We choose the Sraffian supermultiplier (SSM) model of Freitas and Serrano (2015) as the basis of our approach, and we extend the basic SSM model by including employment and productivity. This model already gives a large role to (autonomous) demand in the growth process. We then semi-endogenize productivity with R&D as the main driving factor.[1] The central question that the model tries to address is how, and under which parameter settings, demand, both autonomous (i.e., not dependent on current income) and non-autonomous demand, and productivity growth will adjust to each other to produce a meaningful and stable steady state growth path.

After this introduction, we present a brief literature review in section 2. Section 3 presents our model. Subsections of Section 3 present the baseline model from Freitas & Serrano (2015), our proposal to endogenize (the growth of) autonomous demand and semi-endogenize technological change (productivity growth), the steady state solutions to our model, stability analysis, and several extensions of our main model, including a

[1] By semi-endogenization we mean that productivity growth depends in an indirect way on a parameter which influences the relative accumulation of fixed capital and R&D capital.

rudimentary way to deal with returns to financial investments, and a government stabilization mechanism. In section 4 we summarize the main arguments. More technical details on the model can be found in three appendices.

## 2. A brief review of some relevant literature

The Sraffian supermultiplier (SSM) approach is the core of a demand-led growth model initially proposed by Serrano (1995). We opt for using this idea as the core of our model because of the simplicity of the basic equations of the model: because investment is fully endogenized, the role of demand in growth is reduced to a single parameter, which is the growth rate of autonomous consumption demand. The SSM approach was further developed by Bortis (1996) and De-Juan (2005), and eventually developed into a tradition of models with much recent momentum.

The baseline SSM model of Freitas & Serrano (2015) offered a solution to the issue of Harrodian instability with a constant income distribution. In the Freitas & Serrano (2015) version of the SSM, the capacity utilization rate is a strategic decision of the firms. Firms aim at maintaining a certain degree of idle capacity, allowing them to react under changes in the demand conditions. In the long run, capacity utilization converges to a long-run exogenous rate. The model stabilizes the relationship between productive capacity and aggregate demand by adjustments of the marginal propensity to invest. Because this propensity is an endogenous variable, it enters the multiplier that determines the short-run level of output, hence the term supermultiplier. In this canonical model, income distribution is determined exogenously, given by the institutions that define the bargaining power between wage- and profit-earners, by customs, and by social norms about the fairness of wages.

In the SSM model, investments follow an accelerator mechanism (capital accumulation induced by income) with no autonomous component. Consumption, on the other hand, has an autonomous and an induced (endogenous) component. The autonomous part of consumption grows at an exogenous growth rate. The short-run level of output adjusts to make savings equal to investment ex-post. In this framework, growth is demand-led not only in the short but also in the long run. Economic growth is equal to the exogenous growth rate of autonomous consumption demand, and capital accumulation (given the equilibrium utilization rate) converges to this rate.

The recent literature on the SSM has been focusing on implementing new elements to this basic model, such as income distribution, stock-flow consistency, technological dynamics, and also in proposing distinct alternatives to think about the "non-capacity creating" autonomous component of demand. Work on the latter, i.e., the sources of the exogenous rate of autonomous demand that determines the growth rate of the economy, includes workers' autonomous consumption, financed out of credit (Freitas and Serrano, 2015), as part of the wealth of the workers (Brochier & Silva, 2019), capitalists' consumption (Lavoie, 2016), subsistence consumption including an unemployment benefits system (Allain, 2019), government expenditures (Allain, 2015), exports (Nah & Lavoie, 2017) and R&D investments (Caminati & Sordi, 2019). All these model expansions consider the autonomous expenditure as an exogenous variable, with the exception of Caminati & Sordi (2019), who endogenize it.

Because it is believed that capital is more scarce than labour, and labour ana capital are complementary, the baseline SSM model does not give any attention to employment, with Palley (2019), Brochier (2020) and Fazzari et al. (2020) as notable exceptions. In the long run, and on average, a stable employment rate requires that the growth rate of labour productivity plus the growth rate of labour supply matches the rate of growth of output (we will formalize this in the SSM context in the next section). Economic growth in the SSM converges to the growth rate of the autonomous demand component. However, in the basic SSM model, there are no explicit mechanisms to ensure that this rate is or becomes equal to the sum of labour productivity growth and labour supply (the natural/supply rate of growth).

As stated by Allain (2019), the natural rate of growth and the autonomous expenditure growth rate should be the same, and this adjustment takes place in the labour market. With constant labour productivity, a stable employment rate requires that labour supply reacts to changes in labour demand. Fazzari et al. (2020) and Nah & Lavoie (2019) argue that labour supply growth reacts to the employment rate. The natural rate of growth converges to the autonomous expenditure trend.

Another way for adjustment to take place is through (labour) productivity growth. Fazzari et al. (2020), Nah & Lavoie (2017), and Palley (2019) all propose convergence of labour productivity to the exogenous growth rate. The inclusion of productivity growth is often done by adding a Kaldor-Verdoorn learning effect (Allain, 2019; Brochier, 2020; Deleidi & Mazzucato, 2019), in which, because of learning by doing, productivity growth depends (positively) on the capital accumulation rate or on the growth rate of output. However, with the Kaldor-Verdoorn effect, changes in the exogenous component of productivity growth have only a transitory effect on the capital accumulation rate. Kaldor-Verdoorn is neither a necessary nor a sufficient condition to guarantee employment stability.

Another alternative is presented in Palley (2019). In this approach, with fixed labour supply, labour productivity growth makes the natural growth rate converge towards the autonomous expenditure growth in the long run. That is done by adding a sensitivity element of productivity to employment. When the employment rate increases, productivity growth increases, because of learning effects similar to the Kaldor-Verdoorn mechanism, and when the employment rate falls, the productivity growth rate falls accordingly. Fazzari et al. (2020) who also adopt this particular assumption, discuss a host of justifications for it.

On the other hand, our approach assumes that the labour supply is exogenous and fixed, and does not adjust automatically to demand, which is an assumption that we make to keep the model simple and pure (in terms of the stabilization mechanism). We also assume that productivity growth has its own (R&D-based) dynamics. Thus, in our approach, the burden for stabilization (of employment) falls on the rate of output growth. This means that we will endogenize the rate of growth of autonomous (consumption) demand (following the approach of Brochier & Silva, 2019) and introduce a mechanism (consumption smoothing by workers) that makes this rate adjust to the natural rate of growth.

The SSM model came to the Post-Keynesian debate when non-capacity creating autonomous demand started to be included in endogenous income distribution Neo-Kaleckian models (Lavoie, 2016, Allain 2015). Nah and Lavoie (2019) and Cassetti (2017) include a conflict-theory approach of inflation into the SSM, in which workers and firms negotiate over their shares in GDP, and this affects output prices and wages. Those models, however, consider fixed productivity, which is a problem tackled by Brochier & Silva (2019), who propose including income and wealth distribution using a stock-flow consistent (SFC) framework with endogenous labour productivity. Other efforts to add Stock-Flow consistency (SFC) to the SSM model are in Mandarino et al. (2020) and Brochier (2020). In those models, wealth is distributed between the distinct agents of the system, and consumption out of wealth is included as the key autonomous component. This is also the approach that we will take, albeit in much simplified form, to endogenizing the growth rate of autonomous demand. Cassetti (2017) also deals with cost-induced inflation in a SSM-SFC model, but in his model, instead of household consumption, government expenditures is the autonomous spending component that leads growth.

There have been some efforts to add technical progress (and productivity growth) to the SSM framework. Those efforts have added R&D investments as an autonomous demand element. In this sense, as in Caminati & Sordi (2019), endogenous autonomous expenditure (investment in R&D and consumption) grows in line with productivity. Deleidi & Mazzucato (2019) have proposed the idea of an entrepreneurial state, where government R&D is the autonomous expenditure mechanism.

Our approach to including productivity growth (or technological change) is rooted in the neo-Schumpeterian evolutionary tradition, which mainly looks at technology as a supply phenomenon. This tradition criticizes the static orthodox framework of the Walrasian general equilibrium and proposes a new theory for economic microeconomic dynamics with innovation as a core factor (Nelson and Winter, 1982; Silverberg and Verspagen, 1994). The basis of growth resides in the market implementation of new technologies, in a scenario of competition through innovation. The system follows a process of natural selection, in which the best adapted firms remain in the market.

The Neo-Schumpeterian literature, however, should not only focus on the mechanisms that drive innovation. As mentioned by Hanusch & Pyka (2007), it must also focus on the uncertain developments in the socio-economic system, observing the effects of productive transformation also on other aspects (such as the public and monetary aspects) being "*concerned with the conditions for and consequences of a removal and overcoming of the economic constraints limiting the scope of economic development*." (Hanusch & Pyka, 2007, p.276). In our view, this includes the modelling of the demand side of the economy, i.e., the rejection of Say's law (Meijers et al, 2019). This is the prime motivation for our model that combines (semi-)endogenous technological change (productivity growth) and a (super)multiplier mechanism as the main form of coordination in the economic system. We will explore both the option of household behaviour (in particular, consumption smoothing) and government (fiscal) policy as the key mechanisms that lead to a stable path in a stock-flow consistent version of the SSM model.

## 3. The model

### 3.1. Exogenous growth

Our model follows the Sraffian super-multiplier setup of Freitas & Serrano (2015). We extend their basic model by three variables, after which the model can account for all basic variables of the growth process that we are interested in. Like Freitas and Serrano, we start by modelling investment as a function of the capacity utilization rate:

$$\dot{h} = h\gamma(u - \mu) \tag{1}$$

Here, $\gamma$, the speed of adjustment, and $\mu$, the normal long-run capacity utilization rate, are parameters. $h$ is the share of investment in GDP, and $u$ is the capacity utilization rate. The latter is defined as follows:

$$u = \frac{Y}{Y_K}, Y_K = \frac{K}{v}$$

where $Y$ is output (GDP) $K$ is the stock of fixed capital, $v$ is the normal capital-output ratio, and $Y_K$ is full-capacity output.

An important part of the model is how GDP is determined in the short run. This is a Keynesian multiplier process, in which we distinguish between wage-income and profit-income. As we will discuss in more detail below, the main purpose for making this distinction is to be able to introduce a stabilization mechanism, which will keep the economy on a stable employment path. The consumption function thus makes a distinction between the two sources of income:

$$C = \sigma Y c_w + (1 - \sigma) Y c_p + Z_w + Z_p \tag{2}$$

where $C$ is consumption, the parameter $c$ is a marginal consumption rate, $\sigma$ is the share of wage income in GDP (we will consider this as a parameter, although we would think that further development of the model would endogenize this), $Z$ is autonomous consumption (i.e., independent of current GDP), and the subscripts $w$ and $p$ represent wage income and profit income. We can now derive GDP in the well-known way:

$$Y = C + hY = \sigma Y c_w + (1 - \sigma) Y c_p + Z_w + Z_p + hY \Rightarrow$$

$$Y = \left(Z_w + Z_p\right) \frac{1}{1 - \sigma c_w - (1 - \sigma) c_p - h} \tag{3}$$

where $1/\left(1 - \sigma c_w - (1 - \sigma) c_p - h\right)$ is the supermultiplier.

To make our model resemble that of Freitas and Serrano, we can assume $c_w = c_p \equiv c$ and define $1 - c \equiv s$. We then also drop the distinction between $Z_w$ and $Z_p$, and denote total autonomous consumption by $Z$. Then the supermultiplier becomes $1/(s - h)$ and equation (3) reduces to

$$Y = Z \frac{1}{s - h} \tag{4}$$

Writing the growth rate of $Z$ as $g_Z \equiv \dot{Z}/Z$ (a dot above a variable will denote a time derivative), and still referring to the reduced (Freitas and Serrano) version of the model, we can write the growth rate of GDP in the following way:

$$\frac{\dot{Y}}{Y} \equiv g = g_Z + \frac{\dot{h}}{s-h} \tag{5}$$

This equation allows us to return to the growth context of the model. The original Freitas and Serrano model consists of two equations: (1) and a differential equation for the capital utilization rate $u$. By its definition, the growth rate of $u$ is equal to the growth rate of GDP ($g$ as in 5) minus the growth rate of the capital stock. We denote the latter as $g_K \equiv \dot{K}/K$. Like Freitas and Serrano, we assume that capital accumulation is a perpetual inventory process, with a fixed depreciation rate $\delta$:

$$\dot{K} = I - \delta K = hY - \delta K \Rightarrow g_K = \frac{hu}{v} - \delta \tag{6}$$

With this, we can write the differential equation for the capacity utilization rate:

$$\frac{\dot{u}}{u} = g - g_K = g_Z + \frac{\dot{h}}{s-h} - \frac{hu}{v} + \delta \tag{7}$$

As we are interested not only in the capacity utilization rate but also in employment, we will derive a similar expression for the employment rate, which we will denote by $E$. We will assume a constant labour force and a fixed labour coefficient in the short run, i.e., $a = Y/L$, where $a$ is labour productivity. We will allow for labour productivity to grow over time, and will set $\dot{a}/a \equiv \rho$. Then

$$\frac{\dot{E}}{E} = g - \rho = g_Z + \frac{\dot{h}}{s-h} - \rho \tag{8}$$

If we consider $g_Z$ and $\rho$ as exogenous, we already have a full model that consists of equations (1), (7) and (8), with variables $h$, $u$, and $E$. In the steady state of this model, equation (1) dictates that $u = \mu$ so that $\dot{h} = 0$, and it follows (from equations 7 and 8) that $\dot{u}/u = g_Z - g_K$ and $\dot{E}/E = g_Z - \rho$. This implies that for any steady state values of $u$ and $E$ to exist, we must have

$$g_Z = g_K = \rho \tag{9}$$

In this equation, which is in line with the analysis in Palley (2019) and Fazzari et al. (2020), $g_K$ is endogenous, but (so far) $g_Z$ and $\rho$ remain exogenous. If, like Freitas and Serrano, we set $\rho = 0$, and disregard equation (8) (i.e., do not consider employment to be a variable of interest), then equation (7) solves for $h = (g_Z + \delta)v/\mu$. In this case, $g_K$ will adjust to become equal to $g_Z$.

Equation (9) is also a reinterpretation of the old idea of Harrod instability. It shows that if we wish to have a (positive) steady state value for the rate of (un)employment, then we must have equality of the two exogenous growth rates ($\rho = g_Z$). With equality, $g_K$ will adjust to become equal to $\rho = g_Z$. However, if $\rho \gtrless g_Z$, and with both of these rates exogenous, there is no change of a steady state with both capital utilization and employment constant (and positive).

Because there is no a priori good reason why in a model of exogenous growth $\rho = g_Z$, we proceed to endogenize both these rates. The task is not only to specify how these two rates are equalized, but also whether (and how) the steady state that results from equalization will be stable. Here we do not take the notion of a stable steady state as an approximation of economic reality, as we know well enough that growth paths (including employment rates) are seldom smooth steady states in actual economic history. Instead, we look at the stable steady state that our model looks after as a baseline economic mechanism upon which we must ultimately seek to add turbulence by means of additional economic factors that will remain unspecified in our current analysis.

### 3.2. Semi-endogenizing $\rho$

In the model with semi-endogenous productivity growth, we start by assuming that a share $\tau$ of GDP is spent (out of profit income) on Research and Development (R&D). We consider $\tau$ as an exogenous parameter, but propose that further development of the model would endogenize this parameter.

With R&D included in the model, equation (3) changes to

$$Y = \left(Z_w + Z_p\right)\frac{1}{1-\sigma c_w-(1-\sigma)c_p-\tau-h} \tag{3a}$$

Further, we define the ratio of the R&D-capital stock to the stock of fixed capital as $\Phi \equiv R/K$. R&D-capital (denoted by $R$) evolves as a stock in the same way as fixed capital $K$, with a fixed depreciation rate $\Delta$ (for most of our steady state calculations, we will assume $\Delta = \delta$, i.e., R&D capital and fixed capital depreciate at the same rate, which will simplify the mathematics):

$$\dot{R} = \tau Y - \Delta R \Rightarrow \hat{R} \equiv \frac{\dot{R}}{R} = \frac{\tau u}{\Phi v} - \Delta \tag{10}$$

Using (6) and (10), we get

$$\dot{\Phi} = \Phi\left(\frac{\tau u}{\Phi v} - \Delta - \frac{hu}{v} + \delta\right) = \Phi\frac{u}{v}\left(\frac{\tau}{\Phi} - \Delta - h + \delta\right) \tag{11}$$

Finally, we assume that productivity growth results from R&D, more specifically, from the value of $\Phi$:

$$\rho = \bar{\rho} + \varphi\Phi \tag{12}$$

where $\bar{\rho}$ (the exogenous part of productivity growth) and $\varphi$ are parameters. Note that if we set $\tau = 0$, it follows that $\Phi = 0$, or if we set $\varphi = 0$, we are back in the realm of completely exogenous productivity growth ($\rho = \bar{\rho}$).

### 3.3. Endogenizing $g_Z$: private spending

The first idea that we will employ for endogenizing $g_Z$ is that private autonomous consumption depends on accumulated wealth. This is the foundation of the first (and main) model that we present, while in a later subsection we will consider government fiscal policy as another source of endogenous autonomous spending in the economy in an alternative model.

In the appendix, we present a model where, in line with the general consumption function (2), we distinguish between consumption out of wage income as well as profit income, and accumulated wealth is also from wage income and accumulated wealth from profit income. For clarity in exposition, here, in the main text, we will focus on a special case, where there is no consumption out of profit income, nor out of accumulated wealth out of profits. In other words, only wage income and the accumulated savings out of wage income are used for consumption. In this special case, the basic outcomes of the model in terms of growth are unchanged relative to the more general appendix model, and the formal expressions for steady state values are significantly simplified relative to the general case.

The basic idea for endogenizing $g_Z$ is taken from Brochier (2020), and states that autonomous consumption is dependent on accumulated wealth. In the specific case that we consider here in the main text, this wealth is defined as accumulated savings purely out of labour income (wages). The appendix deals with the more general model in which also accumulated wealth out of profit income is considered.

We denote accumulated savings out of labour income by $W_w$, and specify its motion by the following equation:[2]

$$\dot{W}_w = (1 - c_w)\sigma Y - Z_W - \delta W_w \tag{A2a}$$

The first part of the righthand side ($(1 - c_w)\sigma Y - Z_W$) simply represents savings out of current labour income. The term $-\delta W_w$ represents depreciation of wealth. This arises from the specific setup of the model, explained in more detail in the appendix, in which total accumulated wealth in the economy is equal to the sum of the productive capital stocks (both R&D capital $R$ and fixed capital $K$). In this way, $W_w$ is seen as an entitlement of the holders (wage earners) on the stock $R + K$. The term $-\delta W_w$ is included because the entitlement to $R + K$ will depreciate with the stocks themselves, and we assume, for simplicity, that R&D capital and fixed capital depreciate at the same rate $\Delta = \delta$.

The general model has a corresponding wealth variable $W_p$, which represents assets held by profit earners. As the appendix shows, $W_w + W_p = R + K$. Because of the specific assumptions made here (no autonomous consumption out of $W_p$), we do not need the variable $W_p$ in the exposition in the main text. However, we do have to introduce the variable $x \equiv W_w/(W_w + W_p) = W_w/(K + R)$, which represents the share of wage earners in total wealth of the economy.

[2] Equation numbers starting with A refer to one of the three appendices. These equations are introduced and discussed in some detail in the appendices, in this case the context of the model with generalized consumption function.

The endogenization of $g_Z$ then proceeds by positing

$$Z_w = \zeta_w W_w \tag{A4a}$$

$$Z_p = \zeta_p W_p \tag{A4b}$$

Here $\zeta_w$ is a new variable that represents the (marginal) propensity to consume out of accumulated workers' savings, and similarly $\zeta_p$ is a parameter that represents the marginal propensity to consume out of profit earners' assets. Note that we assume $\zeta_p = 0$ (as well as $c_p = 0$) in the main text, i.e., equation (A4b) is only reported here for completeness (these assumptions are relaxed in Appendix 1). Our assumption is also that the variable $\zeta_w$ is a behavioural variable that serves to smooth (autonomous) consumption spending for changes in workers' income that result from changes in the employment rate. More specifically, we specify

$$\dot{\zeta_w} = \iota \zeta_w (\bar{E} - E) \tag{13}$$

Here $\iota$ and $\bar{E}$ are parameters (both >0). $\bar{E}$ specifies a neutral rate of employment at which current wage income is considered satisfactory. When the employment rate drops below $\bar{E}$, current labour income also falls below the satisfactory level (remember we assume a fixed real wage rate), and workers have to "compensate" by drawing to a larger extent on their accumulated wealth for consumption. This means that $\zeta_w$ will have to rise. Similarly, when employment rises above $\bar{E}$, labour income is considered high, and there is less of a need for consumption out of accumulated wealth. Hence $\zeta_w$ will fall. We adopt the shorthand term "consumption smoothing" (James et al., 2007; Kim et al., 2014) for the idea specified by equation (13), which is a key mechanism in our model that proves to provide stability to the growth path in terms of ensuring a stable employment rate.

It is easy to see how equation (13) has the potential to stabilize the economy. If employment falls below the neutral value ($\bar{E}$), autonomous consumption will tend to increase ($W_w$ will be fixed initially, while $\zeta_w$ increases), and ceteris paribus the multiplier, GDP will increase, bringing the employment back towards the neutral rate $\bar{E}$. Note that such stabilization works exclusively through quantity adjustment (of autonomous demand). Because demand in our model depends largely on consumption out of wage income, it is hard to imagine how flexible prices would achieve stabilization. For example, a traditional Phillips curve regulating (real) wages would have a de-stabilizing effect, because wages (and hence demand) would fall when employment falls below a threshold level, leading to a spiral that bring the economy further away from a stable employment rate.

The ultimate model (with semi-endogenized $\rho$ and fully endogenized $g_Z$) contains six variables: $h$, $u$, $E$, $\Phi$, $\zeta_w$ and $x$. We have already specified differential equations for $h$, $\Phi$ and $\zeta_w$ (equations 1, 11 and 13, respectively). We also have general forms (equations 7 and 8) for the differential equations for $u$ and $E$, in which we still need to specify the endogenized variable $g_Z$. We also need to specify the differential equation for $x$.

Let us start by writing the expression for $g_Z$, which will give us two differential equations. Clearly, from equation (A4a), $g_Z = (\dot{\zeta}_w/\zeta_w) + (\dot{W}_w/W_w)$. The first of the terms on the righthand side of this follows directly from equation (13). With the assumption that autonomous consumption out of profit income is zero, we can also write

$$g_W \equiv \frac{\dot{W}_w}{W_w} = \zeta_w x\left(\frac{\tau+h}{1-c_w\sigma-\tau-h}\right) - \delta \tag{14}$$

(This equation is a specific case of equation A6). This leads to

$$g_Z = \iota(\bar{E} - E) + \zeta_w x\left(\frac{\tau+h}{1-c_w\sigma-\tau-h}\right) - \delta \tag{15}$$

(the more general form of this is A7).

And then:

$$\dot{u} = u\left(\iota(\bar{E} - E) + \zeta_w x\left(\frac{\tau+h}{1-c_w\sigma-\tau-h}\right) + \frac{\dot{h}}{1-c_w\sigma-\tau-h} - \frac{hu}{v}\right) \tag{7'}$$

$$\dot{E} = E\left(\iota(\bar{E} - E) + \zeta_w x\left(\frac{\tau+h}{1-c_w\sigma-\tau-h}\right) + \frac{\dot{h}}{1-c_w\sigma-\tau-h} - \delta - \bar{\rho} - \varphi\Phi\right) \tag{8'}$$

Finally, we can derive the last differential equation from the definition $x = W_w/(W_w + W_p)$:

$$\dot{x} = \frac{(\zeta_w x + \zeta_p(1-x))((1-c_w)\sigma - x(\tau+h))}{1-c_w\sigma-c_p(1-\sigma)-\tau-h} - \zeta_w x \tag{16}$$

### 3.4. Steady state

In this section, we will analyze the steady state of the model with semi-endogenous productivity growth and endogenous $g_Z$ as specified in the previous section (i.e., private autonomous spending as the source of $g_Z$). We will ignore the trivial steady state solution where all variables are zero.

We can start by setting equations (1) and (13) to zero, which immediately yields $u^* = \mu$ and $E^* = \bar{E}$ (we will denote steady state solutions by a * superscript). Next, setting equations (7') and (8') to zero, re-arranging and equating the results of both equations, as well as substituting (15), yields the following quadratic equation for $\Phi$:

$$\varphi\Phi^2 + (\delta + \bar{\rho})\Phi - \frac{\tau\mu}{v} = 0 \tag{17}$$

Obviously, this equation has two solutions, but only one of them yields a positive value for $\Phi$:

$$\Phi^* = \frac{-(\delta+\bar{\rho}) + \sqrt{(\delta+\bar{\rho})^2 + \frac{4\varphi\tau\mu}{v}}}{2\varphi} \tag{18}$$

To derive the steady state value for $h$, two routes are now open. One of these is to set equation (11) to zero, which yields (remember we assume $\Delta = \delta$ for simplicity):

$$\dot{\Phi} = \Phi \frac{u}{v}\left(\frac{\tau}{\Phi} - h\right) = 0 \Rightarrow \frac{\tau}{\Phi^*} = h^* \quad (19)$$

Equation (18) could be substituted into (19) to obtain $h^*$ as a function of only parameters. The other route is to set (7') and (8') to zero, solve each of these for $\zeta_w$ and equate the two expressions, which yields

$$h^* = \frac{v}{\mu}(\delta + \bar{\rho} + \varphi\Phi^*) \quad (20)$$

Again, we can substitute (19) to obtain an (alternative) expression for $h^*$ as a function of only parameters. Equation (20) also shows that the steady state value for the investment rate $h$ depends on the rate of technological change (or productivity growth): as productivity growth rate is faster, capital needs to accumulate at the same rate (see equation 9), which requires a higher value of $h$.

However, rather than actually presenting either one of these expressions (which are equivalent), we show in Figure 1 how the parameter $\tau$ influences the simultaneous determination of the steady state solutions $\Phi^*$ and $h^*$, as well as the resulting rate of productivity growth, $\rho$. The figure shows that all three of these steady state outcomes are concave functions of $\tau$. In other words, increasing the rate of R&D spending as a fraction of GDP will have a positive but declining effect on the steady state values of the investment rate ($h$), the ratio of the R&D stock to the stock of fixed capital ($\Phi$), and, as a result of the latter, the rate of productivity growth.

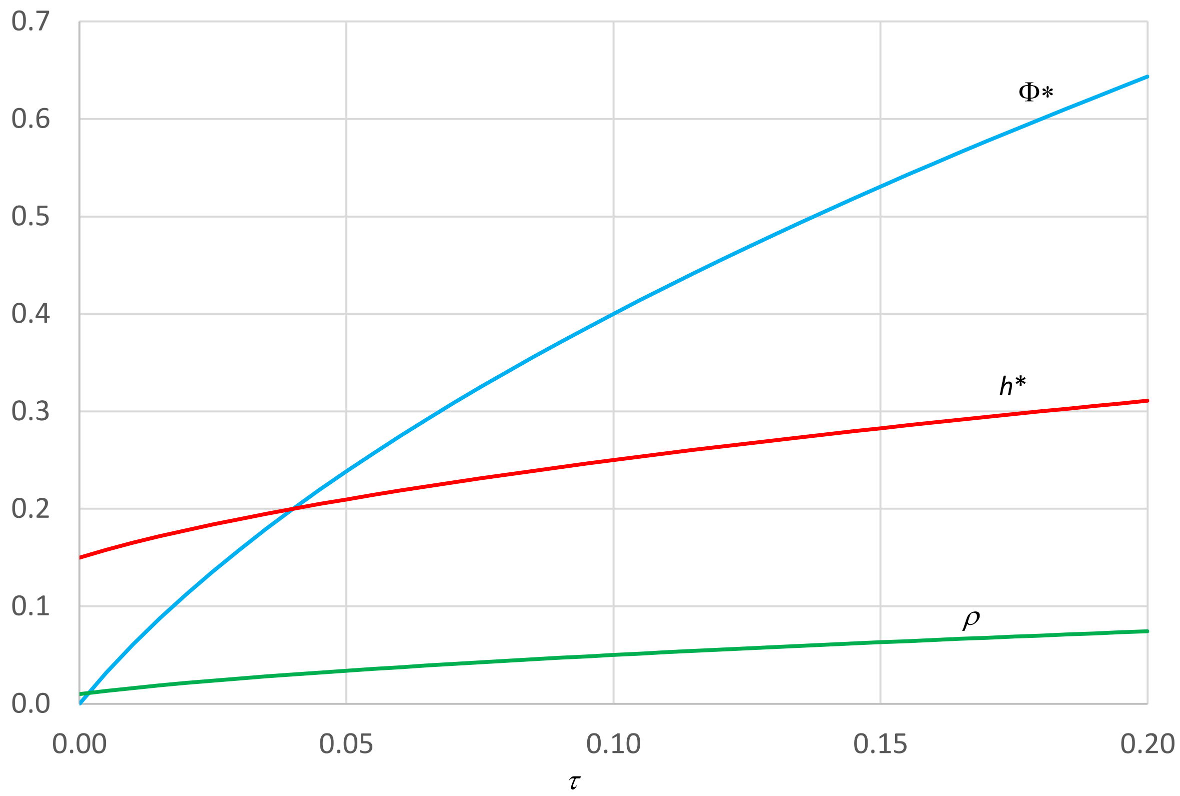



**Figure 1. Steady state values of $\Phi$, $h$ and $\rho$ as a function of $\tau$**

Next, we set equation (16) to zero, solve for $\zeta_w$ and equate this to the expression for $\zeta_w$ that can be derived from setting (8') to zero. This yields the steady state solution for $x$:

$$x^* = 1 - \frac{(1-\sigma)}{(\tau+h^*)} \tag{21}$$

Obviously, the denominator of the fraction on the righthand side $(\tau + h^*)$ is the fraction of GDP that needs to be invested (in fixed capital and R&D) in the steady state. The higher this share is, the more firms need to borrow from workers to fund investment, and hence the higher the steady state value of $x$.

Finally, we can use any of the expressions that were derived for $\zeta_w$ and substitute $x^*$ to obtain

$$\zeta_w^* = \frac{(1-\sigma c_w-(\tau+h^*))h^*\frac{\mu}{v}}{(\tau+h^*)-(1-\sigma)} \tag{22}$$

In this expression, as long as $\tau + h^* < 1$, $\sigma$ has a negative effect on $\zeta_w^*$ (the higher the share of wages in GDP, the lower the resulting rate of autonomous consumption spending). The effect of $\tau$ is also negative, which is in line with our conclusion on equation (21) (the higher $\tau$, the more firms tend to borrow). The effect of $h^*$ is harder isolate, but it appears to be negative, with the same intuition as the effect for $\tau$.

With all steady state values of the endogenous variables derived, we are able to look at how growth works in this model. Equation (9) states that $g_Z = g_K = \rho$, and we also know that in the steady state the growth rate of GDP ($g$) is equal to this. Using equation (12), we see that in the steady state $\rho = \bar{\rho} + \varphi\Phi^*$, and equation (18) shows that $\Phi^*$ depends on a range of parameters, which includes $\delta$, $\tau$, $\mu$, and $v$ ($\bar{\rho}$ and $\varphi$ were already included).

All of these are supply-side parameters, and some ($\tau, \bar{\rho}$ and $\varphi$) are directly related to technological change. Demand-side parameters, such as $c_w$, $\bar{E}$ or even $\sigma$ do not enter the expression for the long-run growth rate of the economy. What happens is that $g_Z$ (the demand side) adjusts to the growth rate of productivity. This also implies that the endogenization of the demand side ($g_Z$) is crucial for the existence and stability of a steady state. In fact, we could keep the rate of productivity growth completely exogenous (e.g., $\varphi = 0$), and, as long as we keep the endogenization of $g_Z$, the steady state of the model would still exist.

In the next section, we will consider whether demand side adjustment can produce a stable path towards the steady state values that we derived. Before we undertake to answer this question, we can also note that the general model, for which we document the steady state expressions in the appendix, arrives at the same conclusion with regard to the unique importance of the supply side in determining the growth rate. In other words, also if we relax the assumption that only wage earners consume, we see no change in the steady state growth rate of the economy. Only the steady state values of $x$ and $\zeta_w$ will change if we relax those assumptions.

### 3.5. Stability analysis

We used numerical simulations to explore the behaviour of the model as specified so far.[3] These simulations were done in R, using the ssmmod function, which numerically integrates the equations. Figure 2 documents the time paths for the variables of the model in the baseline simulation, which uses the following parameter values: $\tau = 0.05$, $\delta = 0.05$, $v = 2$, $\mu = 0.8$, $\bar{E} = 0.8$, $c_w = 0.6$, $\sigma = 0.8$, $\varphi = 0.1$, $\bar{\rho} = 0.01$, $\iota = 0.85$, $\gamma = 0.15$ as well as $c_p = 0$ and $\zeta_p = 0$ which we assumed throughout the main text so far. We see that, for these parameter values, the model converges (with dampened fluctuations) to the steady state.

3 For a more detailed calibration of the baseline SSM, see Haluska et al. (2021) for the US economy..

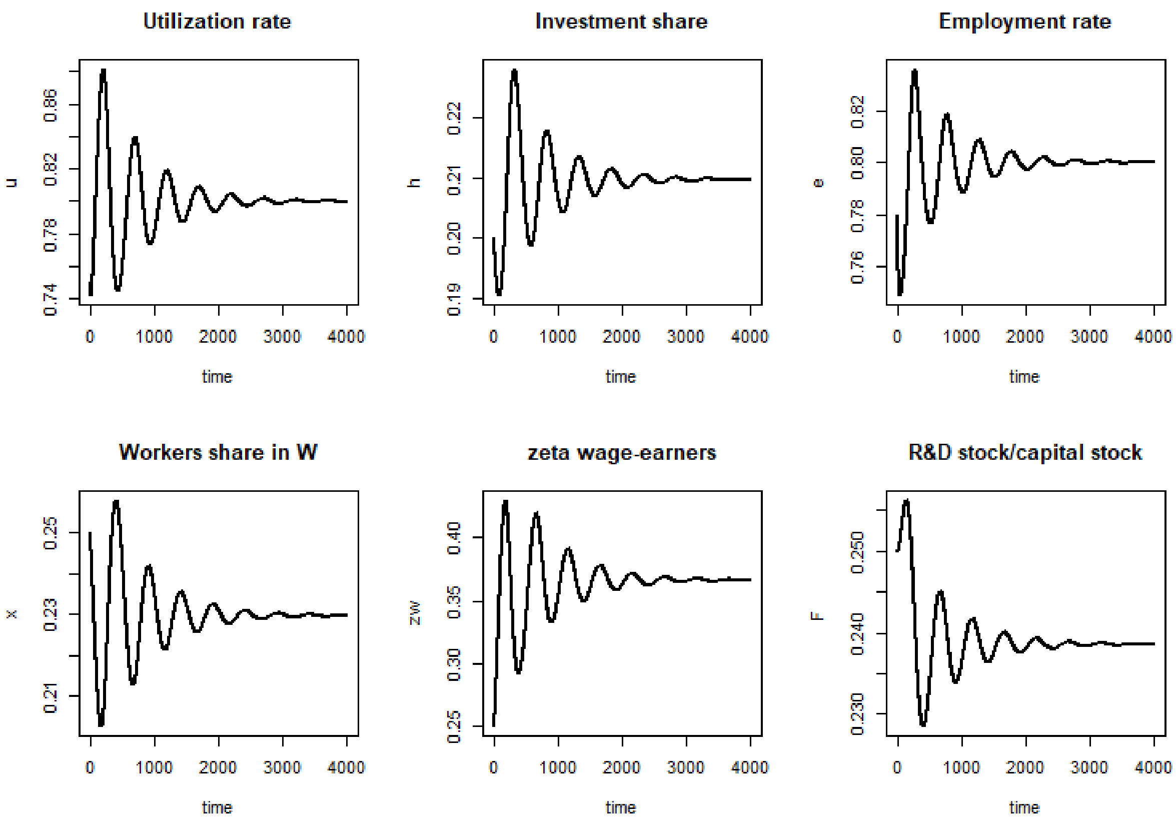



**Figure 2. Model simulation showing stability of the steady state**

In order to obtain a more comprehensive overview of stability, we used Matlab's symbolic math toolbox to derive the Jacobian matrix corresponding to the model. This enables us to do a grid search of parameter space, and numerically calculate the eigenvalues of the Jacobian matrix at the steady state, for each particular parameter constellation. With 11 (or even 13) parameters, it is impossible to do a complete search. Therefore, we limited our search of parameter space to just 5 parameters, which are $\tau$, $\sigma$, $c_w$, $\iota$ and $\gamma$. The other parameters are fixed at the values listed above.

The parameter grid search not only gives information about stability of the steady state. It also provides insights into the sign of some of the steady state values of the variables. In the broad and coarse grid search that we implemented[4], it appeared that there are parameter sets in which either $x$ or $\zeta_w$ (or both) are negative (and stable). While such negative values are not impossible to interpret (essentially, they represent an indebted working class), we will focus on parameter values that yield positive values for $x$ and $\zeta_w$. The parameter grid search suggests that we need fairly high values of $\sigma$ (typically 0.75 or higher for the restricted model of the main text) and $\tau$ (typically 0.05 or larger) to ensure this. Relaxing the assumption that profit earners do not consume reduces the likelihood of negative steady state values for $x$ or $\zeta_w$ considerably. The key parameter to

[4] We analyzed the following ranges in this broad and coarse grid search $\tau$: 0.01 – 0.07; $\sigma$: 0.5 – 0.9; $c_w$: 0.4 – 0.9, $\iota$: 0.01 – 1.5; and $\gamma$: 0.01 – 0.25.

relax this assumption is $\zeta_p$. If we set $\zeta_p = 0.15$, values of $\sigma = 0.7$ or lower still generate positive values for $x$ and $\zeta_w$.

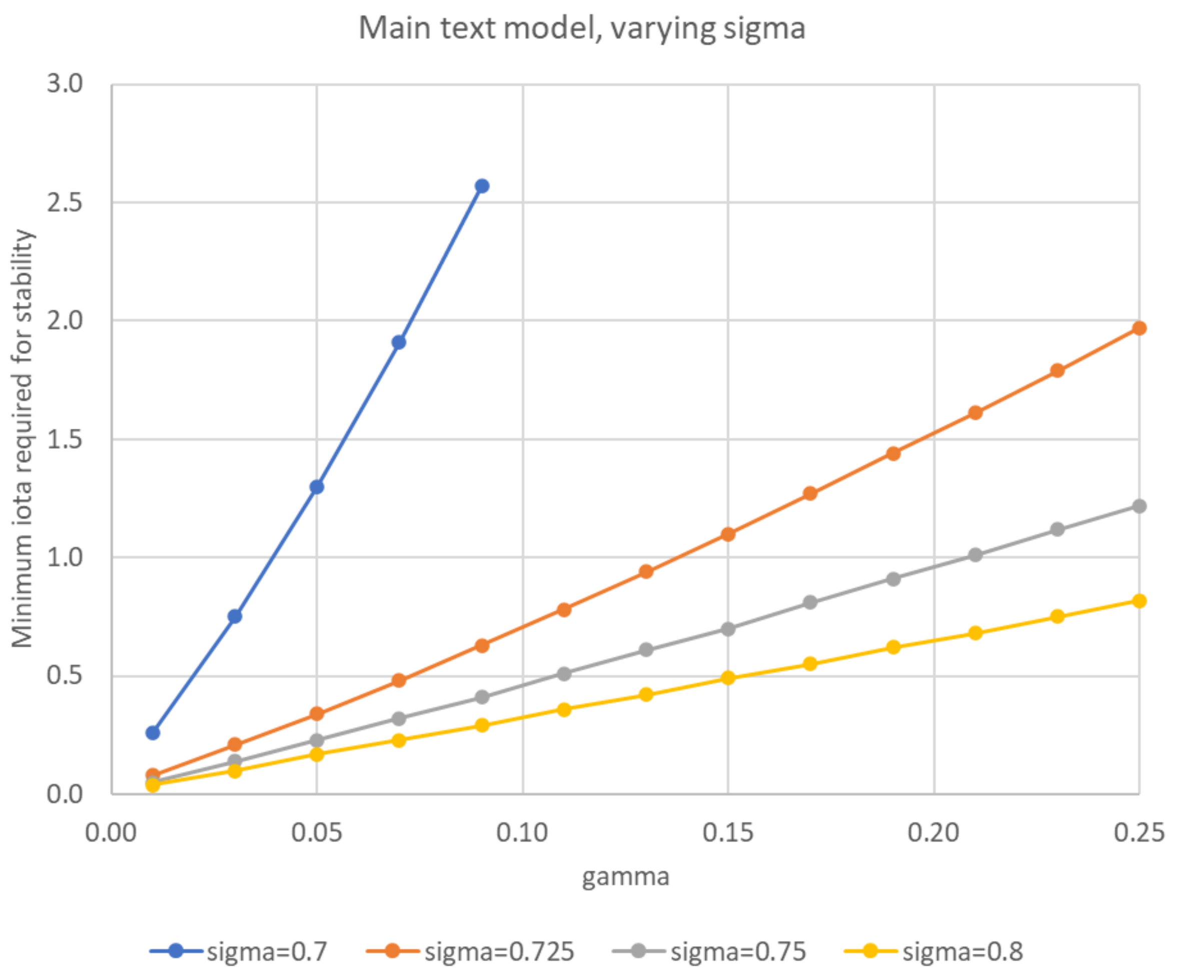



**Figure 3. The role of parameters $\gamma$ and $\iota$ in the main model for stability of the steady state**

Inspection of the results of the grid search suggests that the parameters $\iota$ and $\gamma$ play a crucial role in stability. In particular, we need a minimum value of the ratio $\iota/\gamma$ for the steady state to be stable. This is illustrated in Figure 3, which documents, for different (and fixed) values of the parameters other than $\iota$ and $\gamma$, on the vertical axis the minimum value of $\iota$ that yields a stable steady state given the value of $\gamma$ on the horizontal axis.[5] For example, the grey line (which is drawn for a value $\sigma = 0.75$) shows that if $\gamma = 0.15$, stable steady state values are obtained for values of $\iota \geqq 0.7$. All lines in the figure are (approximately) linear, which means that along each line, the ratio $\iota/\gamma$ is fixed, and we

[5] A stable steady state, in this case means that all eigenvalues (of the Jacobian matrix evaluated at the steady state) are either non-imaginary and negative, or imaginary with a negative real part. We also checked for zero real (parts of) eigenvalues, but this did not happen in the cases we considered.

need this ratio to be larger than the slope of the line for the model to have a stable steady state. This implies that $\iota$ needs to be relatively large for stability, and that larger values of $\gamma$ require larger values of $\iota$.

The figure also provides different lines for varying values of $\sigma$. These indicate that the lower $\sigma$ is (e.g., the blue line represents the lowest value of $\sigma$ that we considered here, 0.7), the higher the required ratio $\iota/\gamma$ is. Given $\gamma$, higher values of pose lower restrictions on $\iota$ for stability. For example, the individual simulation run that we documented above (with $\sigma = 0.8$, $\gamma = 0.15$ and $\iota = 0.85$) lies well above the yellow line.

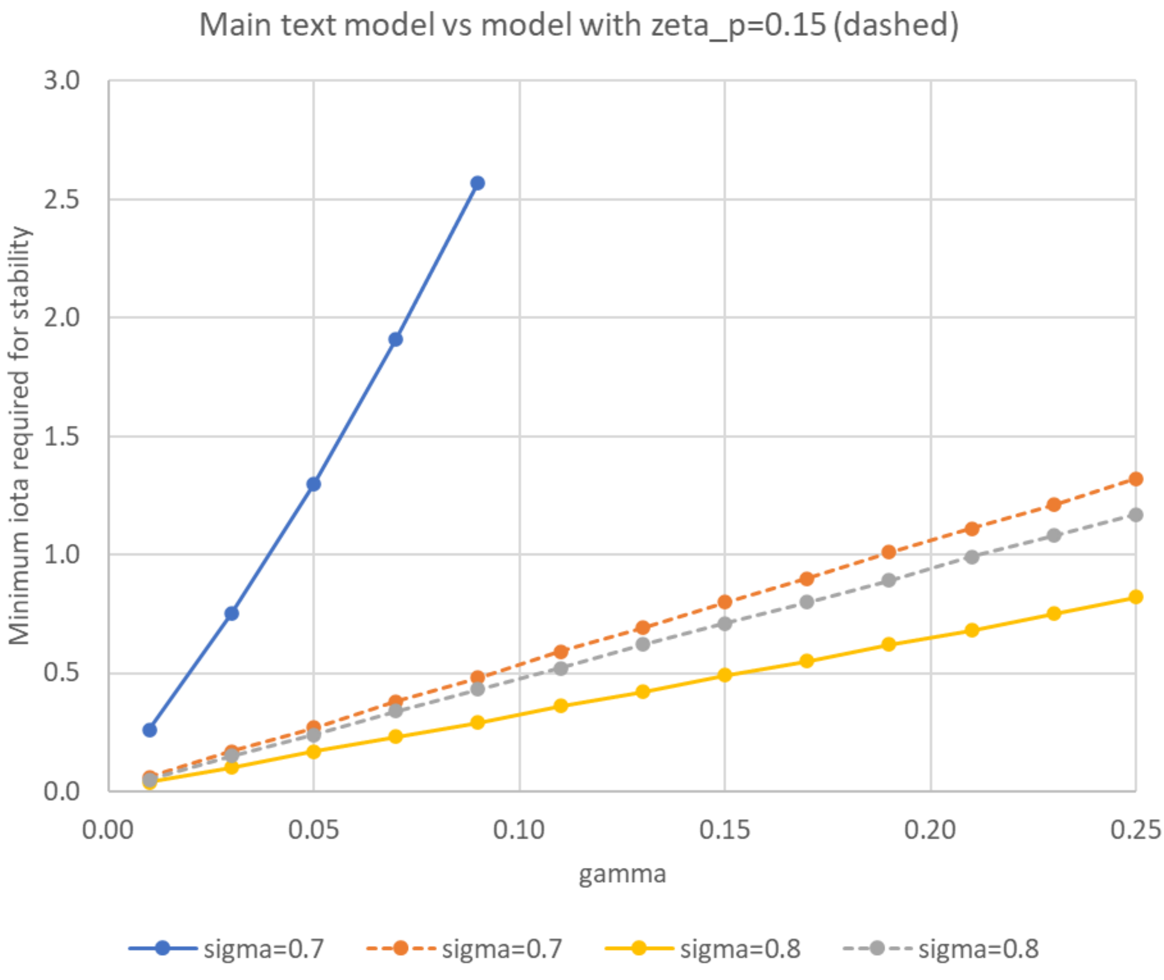



**Figure 4. The role of parameters $\gamma$ and $\iota$ in the main model with zero and positive $\zeta_p$ for stability of the steady state**

Figure 4 documents similar results, but now comparing to a more general case where we relax the assumption $\zeta_p = 0$ and instead set $\zeta_p = 0.15$ ($c_p = 0$ remains as an assumption). Here we only consider the 'extreme' values $\sigma = 0.8$ and $\sigma = 0.9$, i.e., the blue and yellow curves are the same as in the previous figure. Interestingly, compared to the case $\zeta_p = 0$, $\zeta_p = 0.15$ moves the lines $\iota/\gamma$ much closer together. This means that with $\zeta_p = 0.15$, differences in $\sigma$ matter much less for stability than in the case $\zeta_p = 0$. The intuition for this result lies in the variable $x$. Because profit earners now consume out of their wealth, their share of total assets in the economy decreases, i.e., the steady state value for $x$ increases. With a larger base for their autonomous consumption, consumption smoothing becomes easier for workers, which makes $\iota$ a less crucial parameter for stability.

### 3.6. Introducing a rate of return to accumulated savings

So far, we have assumed that there is no return on the accumulated savings by workers. This is, of course, an unrealistic assumption. Ideally, the model that we presented so far would be extended to include a financial market, which would offer various instruments that could be used to invest savings. Such a financial sector would also have to include additional agents, such as a government, a central bank and private banks. While the tradition of so-called stock-flow-consistent models (Godley and Lavoie, 2007; Brochier & Silva, 2019) offers such models, we leave the extension of our model in this elaborate way to future work. Instead, we opt here for a very rudimentary way of incorporating a rate of return.

Our main idea, which we already briefly referenced above, is that accumulated workers' savings can be seen as a claim on the production factors that are accumulated in firms, i.e., fixed capital and knowledge capital (R&D). Remember that total assets (aggregated over workers and profit earners) in the economy are equal to the sum of the capital stock and the R&D stock. Workers' assets (accumulated savings) are a share of this (which is the variable $x$), and our equations will specify that workers are entitled to a proportional share of current profits.

Such an allocation of part of profit income to workers means that workers now receive more (if $x > 0$) than the share $\sigma$ of GDP. If this redistribution would be completely proportional, workers would receive a share $\sigma + x(1-\sigma)$, and profit earners a share $(1-x)(1-\sigma)$. However, we introduce a new parameter, $0 \leq r \leq 1$, which measures the extent to which profits are redistributed. With the inclusion of this new parameter, workers will receive a share $\sigma + rx(1-\sigma)$ of GDP, and profits earners a share $(1-\sigma)(1-rx)$. It is easily seen that if $r = 0$ (i.e., no returns on accumulated savings), we have the model as it has been presented so far, while if $r = 1$, we have the expressions as firstly introduced in this paragraph (returns fully proportional to $x$).

The marginal propensities to consume (or save) for workers and profit earners are applied to total income, i.e., to the shares of GDP as specified in the previous paragraph. This changes some of the equations in the model, and this is documented in full detail in Appendix 2. Here, we only summarize the main result of this change, which solely lies in the steady state values for the variables $x$ and $\zeta_w$. The steady state expressions are given in the appendix. Generally, for $r > 0$, we find larger steady state values for $x$ as compared to equation (21), or its more general counterpart found in Appendix 1 (equation A12). For the case $r = 1$, we find $x^* = 1$, i.e., workers own the entire capital stock. For all cases $r < 1$, $x^* < 1$ remains, thus as long as profit-earners are left with some of the current-period profits, they will always be able to accumulate at least some level of positive assets. However, when $r = 1$, and independently of how much they save, the savings of the profit earners become insufficient to compensate for the depreciation on the physical assets they hold.

The positive effect on $x^*$ in cases $r > 0$ has implications for stability. This is shown in Figure 5, which is similar to the two previous figures. Here all lines are drawn for $\sigma = 0.8$. The solid line is the case of the model of the main text, i.e., $\zeta_p = 0$, which yields the same line as already shown in the previous two figures. The dashed grey line assumes $\zeta_p = 0.15$ but keeps $r = 0$, and hence this is the same line as in the previous figure. The other two lines introduce two new cases: $r = 0.5$ (green) and $r = 1$ (red). We can see that a higher value of $r$ shifts the $\iota/\gamma$ tradeoff line down, i.e., increases stability. This is the same effect as observed before: the increase in $x^*$ makes consumption smoothing easier to implement for given $\iota$.

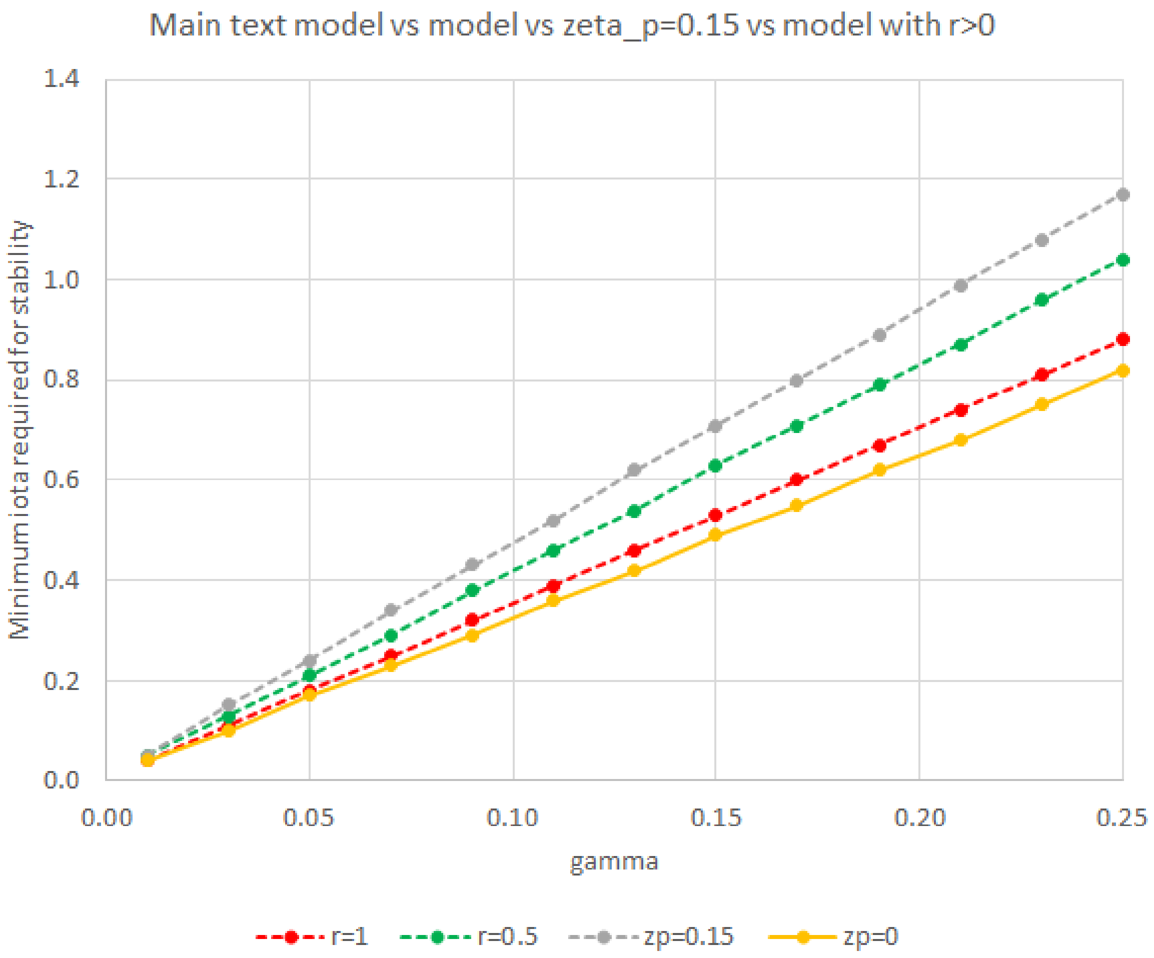



**Figure 5. The role of parameters $\gamma$ and $\iota$ in the model with a financial return for stability of the steady state**

### 3.7. Endogenizing $g_Z$: government spending

We conclude our model analysis by considering an alternative mechanism for stabilizing the economy. Whereas so far private autonomous consumption spending worked as a stability mechanism by workers' desire to smooth consumption between periods of high and low unemployment, we now ask whether the government is able to stabilize the economy by fiscal policy. Here, we will discuss the changes that are made to the model of the previous sections to analyze this question. Full details of some of the equations of the model with government fiscal policy are provided in Appendix 3.

In order to consider the model with a government, we will make one major simplification to the model as considered so far: we will no longer distinguish between workers and profit earners in the private sector. This means that the variable $x$ is no longer relevant, and that we have only a single parameter for the marginal consumption rate (we will denote this parameter by $c$), and a single variable for autonomous spending by the private sector (this variable will be denoted as $Z_h$). As before, $Z_h$ will be a fraction (denoted by $\zeta_h$) of total private-sector assets (or wealth). Because our focus is on the government sector as a stabilization mechanism, we will assume that $\zeta_h$ is a fixed parameter.

On the other hand, the introduction of a government sector also means that we have to introduce new variables and equations into the model. The first of these variables is $Z_G$, which is autonomous government (consumption) spending. Another variable is the tax rate $T$, which is specified as a share of GDP, which implies that $TY$ is total tax revenue. We also specify total outstanding government debt, which we denote by $G$.

In line with the previous section, we assume that the government has to pay interest on the bonds that it issues to fund outstanding debt $G$. For simplicity, we assume that this rate of return is equal to the private rate of return on invested capital (R&D capital $R$ and fixed capital $K$). We then consider profit income as the return on invested capital, which means that the rate of return is equal to $(1-\sigma)Y/W$, where $\sigma$ is, as before, the parameter that represents the share of wages in GDP, and hence $(1-\sigma)$ is the share of profits. $W$ denotes the total invested capital by private agents, which in the model of the previous sections was split into $W_w$ and $W_p$. Because we have now assumed a fixed $\zeta_h$, we have $W = W_w + W_p = R + K$, where $W_w$ and $W_p$ are variables that are only relevant for the comparison with the model of the previous sections.

Finally, we introduce a new variable $D \equiv G/W$, which is government debt as a share of $W = R + K$. With the interest rate on government bonds equal to $(1-\sigma)Y/W$, total interest payments (to the public, which holds the bonds) are equal to $G\,(1-\sigma)Y/W = (1-\sigma)DY$. Then we have

$$\dot{G} = Z_G + (1-\sigma)DY - TY = Z_G + Y\big((1-\sigma)D - T\big) \tag{23}$$

and

$$\dot{D} = \zeta_G + \frac{[(\zeta_H+\zeta_G)+\zeta_H D][D(1-\sigma-h-\tau)-t]}{1-c(1-t+(1-\sigma)D)-(h+\tau)} + D\delta \tag{24}$$

The government model needs a number of behavioral equations for the government. First, we assume that government spending is proportional (by $\zeta_G$) to the private wealth variable $W$:

$$Z_G = \zeta_G W \tag{A4d}$$

We then also need a behavioral rule for the spending fraction $\zeta_G$:

$$\dot{\zeta}_G = \zeta_G \iota_G (\bar{E} - E) \tag{13a}$$

This is similar to the rule that we used for workers' consumption smoothing in the previous sections (equation13), but here it is the government who takes this task upon itself. The main difference is that the government needs to borrow money to perform this function, whereas workers could draw on their savings. Hence the government raises taxes to fund its debt, and therefore we need a behavioral rule for the tax rate. Here, we will assume that the government sets a long-run neutral value for the variable $D$, and adjusts the tax rate to maintain this value (in the long run):

$$\dot{T} = T\eta(D - \bar{D}) \tag{25}$$

In what follows, we will set $\bar{D}$ (the neutral $D$ value) to zero, which means that the government aims to have no debt in the long run (the model that we analyze is a balanced budget supermultiplier model). This is a strict assumption, which we make for mathematical convenience, but assuming $\bar{D} > 0$ does not fundamentally change the conclusions.

This concludes the model with government stabilization. The model consists of seven endogenous variables: $h$, $\Phi$, $u$, $E$ (all of which were present in the model without a government), $T$, $D$, and $\zeta_G$. The differential equations for $h$ and $\Phi$ are unchanged, they are (1) and (11), respectively. The differential equations for $u$ and $E$ are slightly changed and are specified by substituting the new expression for $g_Z$ (equation A7a in Appendix 3) into the generic forms (7) and (8). These two equations are documented as (7a) and (8a) in Appendix 3. Finally, equations (25), (24) and (13a) provide the differential equations for the new variables $T$, $D$, and $\zeta_G$.

### 3.8. Steady state and stability

The (non-trivial) steady solution of the model can be derived in the same way as was done for the model without a government. The expressions for $h^*$, $\Phi^*$, $u^*$ and $E^*$ do not change from what we had without a government, which leaves only the three new government-related variables. We leave details of the derivations of the steady state values of these variables to the interested reader, and just document these values:

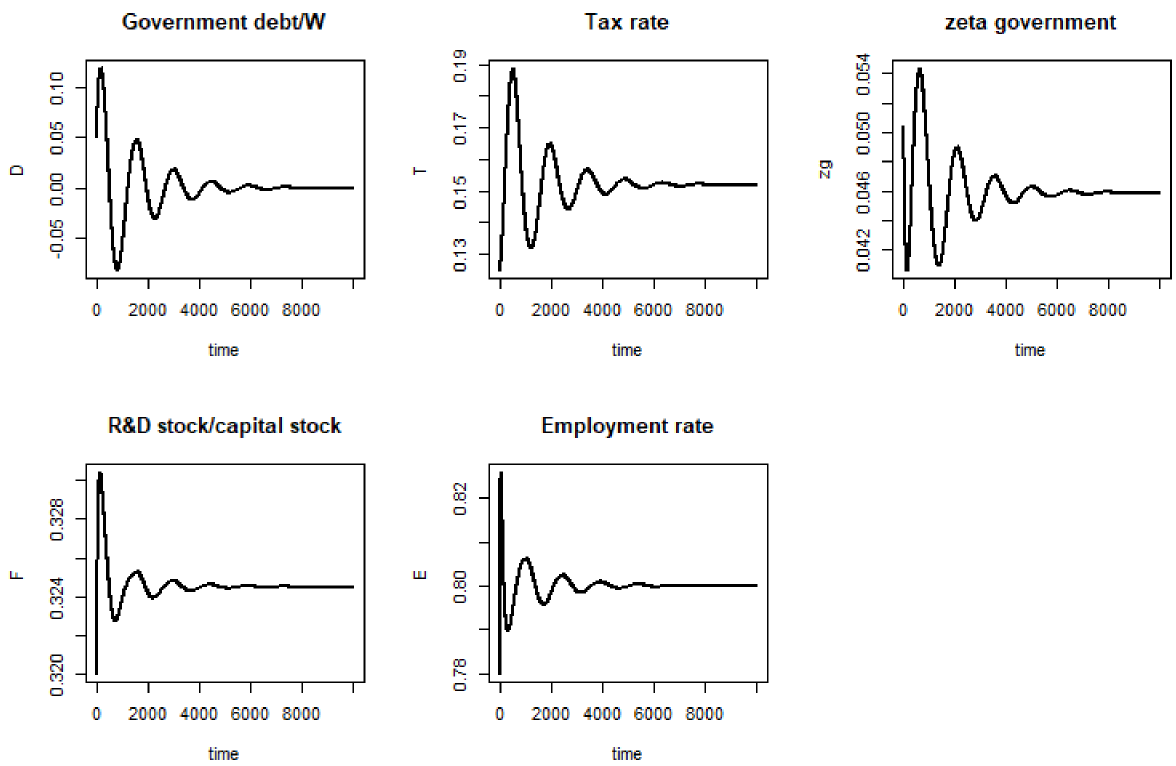
$$D = \bar{D} = 0 \tag{26}$$

$$\zeta_G^* = \frac{\mu}{v(1+\Phi^*)} - \frac{(\delta+\bar{\rho}+\varphi\Phi^*)+\zeta_h}{(1-c)} \tag{27}$$

$$T^* = 1 - \frac{v(1+\Phi^*)}{\mu}\frac{(\delta+\bar{\rho}+\varphi\Phi^*)+\zeta_h}{(1-c)} \tag{28}$$

We simulate this model with the following parameter values: $\tau = 0.075$, $\delta = 0.05$, $\upsilon = 2$, $\mu = 0.8$, $\bar{E} = 0.8$, $c = 0.6$, $\sigma = 0.7$, $\varphi = 0.1$, $\bar{\rho} = 0.01$, $\iota_G = 0.95$, $\gamma = 0.1$, $\zeta_h = 0.01$, and $\eta = 0.1$. Figure 6 shows how, under these parameter values, the model converges to the steady state with dampening oscillations. Along the adjustment path, the variable $D$ also takes negative values, i.e., at some times, the government borrows money from the private sector.

Note that in the set of parameter values that we chose, $\iota_G$ is quite a bit larger than $\eta$. It makes intuitive sense that this is a condition for government fiscal policy to be an effective stabilizer. Obviously, the primary stabilization mechanism is $\zeta_G$, which adjusts in response to (un)employment (equation 13a). On the other hand, equation (25), which determines the dynamic path of the tax rate, works against such stabilization, because a higher tax rate will decrease private consumption. Thus, any positive effects on GDP and employment from increasing $\zeta_G$ will be offset by an increasing tax rate. If the tax effect is immediate, fiscal policy will become ineffective in stabilizing the economy.



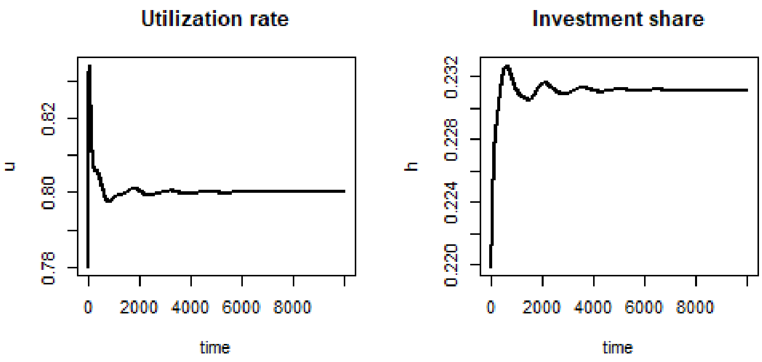



**Figure 6. Government model simulation showing stability of the steady state**

The stability of the steady state in the model with a government seems more precarious than the model without a government that was presented above, i.e., government stabilization using fiscal policy is harder than with the private consumption smoothing stabilizer in the previous sections. One of the reasons for this is that whereas before we had two adjustment parameters ($\gamma$ and $\iota$) for which we needed particular values, we now have three such adjustment parameters: $\eta$, $\gamma$ and $\iota_G$ (and we also have $\overline{D}$, which we fixed at zero for mathematical convenience). Figure 7 presents 2D stability diagrams for each combination of two of these parameters. The underlying data for these diagrams is calculated in the same way as for the previous model, i.e., using Matlab's symbolic math module. In the next diagram, we fix the following parameters: $\tau = 0.05$, $\delta = 0.05$, $v = 2$, $\mu = 0.8$, $\bar{E} = 0.8$, $c = 0.6$, $\sigma = 0.7$, $\varphi = 0.1$, $\zeta_h = 0.02$ and $\bar{\rho} = 0.01$.

Focusing on $\gamma$ and $\iota_G$ first (upper-left corner of the figure), we see that for low values of $\gamma$, the model is always stable, irrespective of the value of $\iota_G$, but note that $\eta$ is fixed at 0.25 (which is a fairly low value) in this diagram. Also high values of $\gamma$ yield a stable steady state, but intermediate $\gamma$ values require a high value for $\iota_G$ for the steady state to be stable. For the combination $\eta$ and $\gamma$ (in the upper-right corner), we need at least one of these two parameters to have a low value (but note that $\iota_G$ is fixed at 0.95, which is a fairly high value). Finally, for the combination of $\eta$ and $\iota_G$ we see that either low or high values of $\eta$ yield a stable steady state, but for intermediate values of $\eta$, we require high values of $\iota_G$ (here $\gamma$ is fixed at 0.1, which is fairly low).

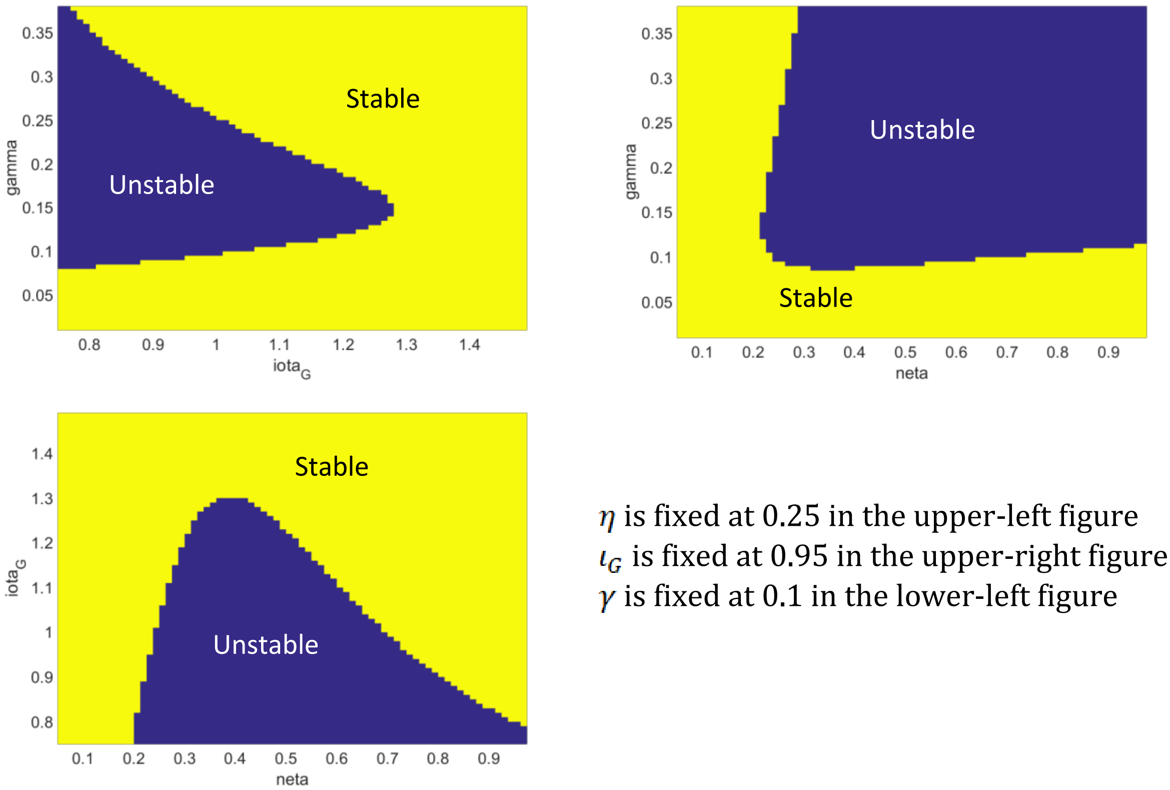



**Figure 7. The role of parameters $\eta$, $\iota_G$ and $\gamma$ in stability of the steady state in the government model**

The rate of technological change and in particular the value of the R&D parameter $\tau$ and the private propensity to consume out of wealth ($\zeta_h$) also have an impact on the stability of the steady state. This is shown in Figure 8, which graphs stability in the $\tau$ vs. $\zeta_h$ plane, and where we fix $\delta = 0.05$, $v = 2$, $\mu = 0.8$, $\bar{E} = 0.8$, $c = 0.6$, $\sigma = 0.7$, $\varphi = 0.1$, $\eta = 0.1$, $\iota_G = 0.95$, $\gamma = 0.01$ and $\bar{\rho} = 0.01$.

Here we see a narrow band of stability emerging, which depicts a tradeoff between the two parameters. High values of one of these parameters require low values of the other for the steady state to be stable, and vice versa. A combination of high $\zeta_h$ and high $\tau$ also yields negative values for the steady state of $T$ and $\zeta_G$, as can be seen in equations (27) and (28), and which are hard to interpret economically. Thus, the upper-right corner of instability in the figure corresponds to this counter-intuitive situation of negative taxes. The lower-left corner of instability corresponds to an economy with low growth rates (due to low R&D investment) and also low private autonomous spending.

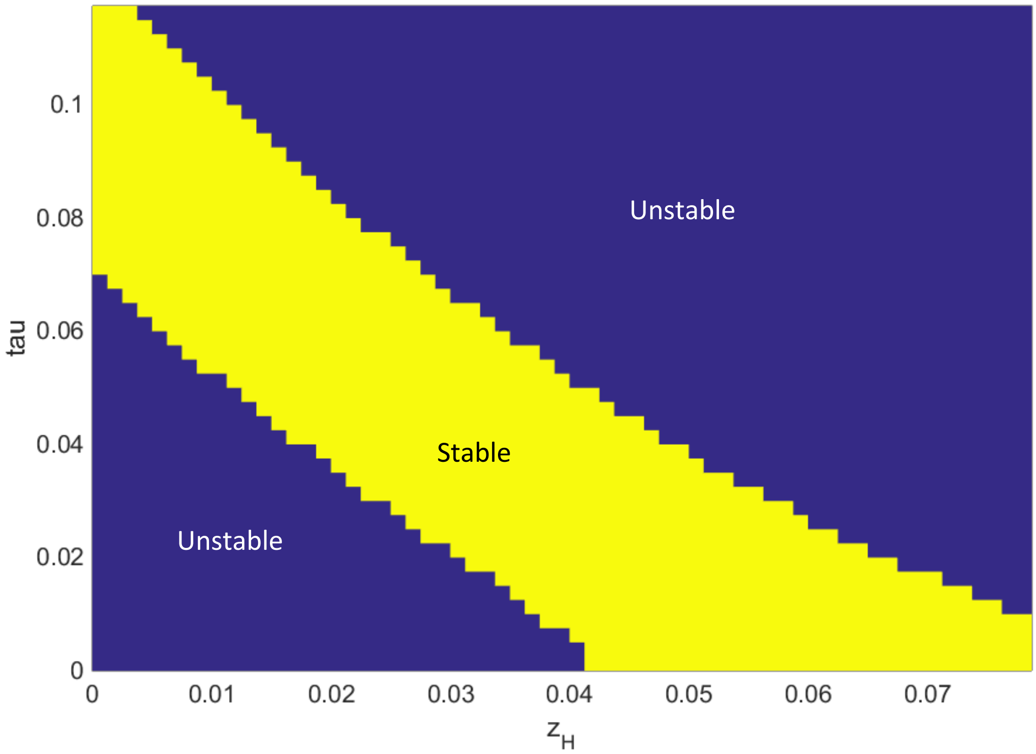



**Figure 8. The role of parameters $\tau$ and $\zeta_h$ in stability of the steady state in the government model**

## 4. Conclusions

The models that we developed in this paper confirm that a long-run steady state with economic growth generated by technological change and endogenous autonomous consumption spending can be consistent with a stable employment rate. As we outlined above, and with our assumption of fixed labour supply, in order to have a stable long run employment rate, autonomous consumption needs to grow at the same rate as labour productivity. Our proposal is that the equality of these two rates is obtained by consumption smoothing by wage earners (workers), who adjust their autonomous consumption spending as a fraction of their accumulated savings, in response to unemployment, and/or by government fiscal policy, where the government runs a temporary deficit (surplus) if the unemployment rate is high (low) and raises taxes to keep its long-run debt within bounds.

The (numeric) analysis of the Jacobian matrix of our models has shown that both these stabilization mechanisms (private consumption smoothing and government fiscal policy) will keep the growth path stable, provided that certain parameter restrictions are satisfied. In the case that stabilization takes place by workers' consumption smoothing, the ratio of the responsiveness of workers' autonomous spending to unemployment to the responsiveness of firms' investment to capital utilization must be large enough. What this minimum value is, exactly depends on parameters in the model, such as the share of wages in GDP, R&D investment as a fraction of GDP, and the rate of autonomous consumption spending by profit earners.

In the model with government fiscal policy as the stabilization mechanism, stability is more difficult to obtain. In this model, various parameters, including the adjustment parameters that govern fiscal policy, but also the responsiveness of firms' investment to capital utilization, as well as the R&D intensity parameter and the propensity of the private sector to consume out of wealth, are crucial parameters that determine stability of the steady state.

Our model variety without a government includes a general consumption function, which allows for consumption spending by workers and by profit owners, and both autonomous consumption (not related to current income) and non-autonomous. Assuming that no consumption (autonomous or otherwise) is done out of profit income simplifies the steady state expressions for the variables in our model, but does not change any of the basic conclusions about growth or stability of the growth path. Also, while most of the time we assume that accumulated workers' savings earn no return, an alternative (and rudimentary) way of modelling such returns suggests that the growth path is unaffected by this (although the distribution of wealth between workers and profit earners is affected). In the model with government fiscal policy, our simplifying assumption is that the government strives for its long-run debt to be zero, and we disregard monetary policy.

In the resulting model, both productivity growth and the growth of autonomous demand indeed appear to be crucial for the emergence of a stable growth path in which we also have stable employment. Technological change (which is modelled by semi-endogenous R&D investments) relieves the resource constraint that the size of the labour force imposes on the economy, and hence makes it possible to achieve per capita growth. Endogenous demand, including endogenous autonomous consumption, keeps the economy on a path where the labour resource is used (at a fixed rate), so that the opportunities provided by technological change are actually utilized.

We feel that there are two main directions in which our model should be extended in future work. On the one hand, while we fully endogenized consumption demand, technological change was only semi-endogenized. Thus, while we considered R&D investment as a fraction of GDP as a fixed parameter, there is scope to consider it as an endogenous variable. This could be done both by making R&D dependent on other macroeconomic variables, such as (expected) profits (as in the endogenous growth literature, e.g., Aghion & Howitt, 1992), or by a behavioral approach that considers R&D at the firm level as resulting from imitation and behavioral mutation (as in Silverberg and Verspagen, 1994). Government spending may also be crucial in the field of technological change (as also in, e.g., in Deleidi & Mazzucato, 2019).

On the other hand, the introduction of a more detailed way of modelling the financial sector would also enhance the degree of realism of the model. This would not only allow the modelling of the (de-)stabilizing effects of finance, but also the inclusion of monetary policy by the government.

## References


Aghion, & Howitt, P. (1992). A Model of Growth through Creative Destruction. *Econometrica*, *60*, 323–351.

Allain, O. (2015). Tackling the instability of growth: A Kaleckian-Harrodian model with an autonomous expenditure component. *Cambridge Journal of Economics*, *39*, 1351–1371.

Allain, O. (2019). Demographic growth, Harrodian (in)stability and the supermultiplier. *Cambridge Journal of Economics*, *43*, 85–106.

Blecker, R.A & M. Setterfield (2020). Heterodox Macroeconomics. Models of Demand, Distribution and Growth, Cheltenham: Edwar Elgar.

Bortis, H. (1996). *Institutions, Behaviour and Economic Theory: A Contribution to Classical-Keynesian Political Economy*. Cambridge University Press.

Brochier, L. (2020). Conflicting-claims and labour market concerns in a supermultiplier SFC model. *Metroeconomica*, *71*, 566–603.

Brochier, L., & Silva, A. C. M. e. (2019). A supermultiplier Stock-Flow Consistent model: The "return" of the paradoxes of thrift and costs in the long run? *Cambridge Journal of Economics*, *43*, 413–442.

Caminati, M., & Sordi, S. (2019). Demand-led growth with endogenous innovation. *Metroeconomica*, *70*, 405–422.

de-Juan, Ó. (2005). Paths of accumulation and growth: Towards a Keynesian long-period theory of output. *Review of Political Economy*, *17*, 231–252.

Deleidi, M., & Mazzucato, M. (2019). Putting Austerity to Bed: Technical Progress, Aggregate Demand and the Supermultiplier. *Review of Political Economy*, *31(3)*, 315-335.

Fazzari, S. M., Ferri, P., & Variato, A. M. (2020). Demand-led growth and accommodating supply. *Cambridge Journal of Economics*, *44*, 583–605.

Freitas, F., & Serrano, F. (2015). Growth Rate and Level Effects, the Stability of the Adjustment of Capacity to Demand and the Sraffian Supermultiplier. *Review of Political Economy*, *27*, 258–281.

Godley, W., & Lavoie, M. (2007). Monetary Economics: An Integrated Approach to Credit, Money, Income, Production and Wealth. In *Investigación económica / Escuela Nacional de Economía, Universidad Nacional Autónoma de México* (Vol. 66). https://doi.org/10.1057/9780230626546

Haluska, G., Braga, J., & Summa, R. (2021). Growth, investment share and the stability of the Sraffian Supermultiplier model in the U.S. economy (1985–2017). Metroeconomica. *Forthcoming.*

Hanusch, H., & Pyka, A. (2007). The Principles of Neo-Schumpeterian Economics. *Cambridge Journal of Economics*, *31*, 275–289.

James, J.A., Palumbo, M.G and M. Thomas (2007). Consumption smoothing among working-class American families before social security. Oxford Economic Papers, vol. 57, pp. 606-640.

Kaldor, N. (1957), A Model of Economic Growth. The Economic Journal, 67, pp. 591-624.

Kim, Y.K., Setterfield, M. and Y. Mei (2014). A theory of aggregate consumption. *European Journal of Economics and Economic Policies: Intervention*, vol. 11, pp. 31–49

Lavoie, M. (2016). Convergence Towards the Normal Rate of Capacity Utilization in Neo-Kaleckian Models: The Role of Non-Capacity Creating Autonomous Expenditures. *Metroeconomica*, *67*, 172–201.

Maddison, A. (1991). *Dynamic Forces in Capitalist Development*, Oxford University Press

Mandarino, G. V., Santos, C. H. D., & Silva, A. C. M. e. (2020). Workers' debt-financed consumption: A supermultiplier stock–flow consistent model. *Review of Keynesian Economics*, *8*, 339–364.

Meijers, H., Nomaler, Ö., & Verspagen, B. (2019). Demand, credit and macroeconomic dynamics. A micro simulation model. *Journal of Evolutionary Economics*, *29*, 337–364.

Mitchell, W., Wray, L. R., & Watts, M. (2019). *Macroeconomics* (1st ed. 2019 Edition). Red Globe Press.

Nah, W. J., & Lavoie, M. (2017). Long-run convergence in a neo-Kaleckian open-economy model with autonomous export growth. *Journal of Post Keynesian Economics*, *40*, 223–238.

Nah, W. J., & Lavoie, M. (2019). The role of autonomous demand growth in a neo-Kaleckian conflicting-claims framework'. *Structural Change and Economic Dynamics*, *51*, 427–444.

Nelson, R., & Winter, S. (1982). *An evolutionary theory of economic change*. Cambridge, Mass. and London: Belknap Harvard.

Pasinetti, L. (1981). Structural Change and Economic Growth. A Theoretical Essay on the Dynamics of the Wealth of Nations, Cambridge: Cambridge University Press.

Palley, T. (2019). The economics of the super-multiplier: A comprehensive treatment with labor markets. *Metroeconomica*, *70*, 325–340.

Serrano, F. (1995). Long period effective demand and the Sraffian supermultiplier. *Contributions to Political Economy*, *14*, 67–90.

Silverberg, G. & B. Verspagen (1994). Learning, Innovation and Economic Growth. A Long Run Model of Industrial Dynamics. *Industrial and Corporate Change*, 3, 199-224.

## Appendix 1. The model with generalized consumption equation

Our model with generalized consumption (and savings) behaviour starts from equation (2) in the main text, which is the consumption function:

$$C = \sigma Y c_w + (1-\sigma) Y c_p + Z_w + Z_p \quad (2)$$

Next we define savings as any income that is not consumed, and we distinguish between savings out of labour income and out of profit income. Savings out of labour income are $S_w = \sigma Y - \sigma Y c_w - Z_w$, while savings out of profit income are $S_p = (1-\sigma)Y - (1-\sigma)Y c_p - Z_p$. Note that, by the usual identity, total savings are equal to total investment (R&D and fixed capital):

$$Y = \sigma Y c_w + (1-\sigma) Y c_p + Z_w + Z_p + hY + \tau Y \Rightarrow S_w + S_p = (h+\tau)Y \quad (A1)$$

We assume that total current investment $(h+\tau)Y$ accumulates into a stock that is held by profit-earners. By slightly re-writing equation (A1) to

$$S_w = (h+\tau)Y - S_p \quad (A1')$$

we see that savings from labour income are matched by the excess of investment over savings from profit income. Although it is possible that total investment is smaller than savings from profit income, most of our analysis will focus on the case where savings out of labour income are positive, and hence investment exceeds savings out of profit income.

This implies that workers build up positive assets, and that these assets represent holdings on the profit-earning class (firms). Firms, however, also build up assets, which are the means of production (R&D capital and fixed capital), hence workers' assets represent holdings on these means of production. But capital also depreciates, which diminishes the value of the total assets of labour and profit earners together. We choose to attribute depreciation to both workers and profit earners, in proportion to total (net) assets held by each class.[6]

As was done in the main text, we assume (for mathematical convenience) that R&D capital and fixed capital depreciate at the same rate $\delta = \Delta$. This leads to the following equation for the accumulation of assets held by workers, which we denote by $W_w$:

$$\dot{W}_w = (1-c_w)\sigma Y - Z_W - \delta W_w \quad (A2a)$$

The corresponding assets held by profit earners are denoted by $W_p$, and these accumulate according to

$$\dot{W}_p = \left(1-c_p\right)(1-\sigma)Y - Z_p - \delta W_p \quad (A2b)$$

Now with $Y = (Z_w + Z_P)/(1 - c_w\sigma - c_P(1-\sigma) - \tau - h)$, equations (A2a) and (A2b) imply

[6] Alternative assumptions are possible, but make the mathematics more involved. Generally, making different assumptions about how depreciation of wealth is handled only affects the steady state distribution of wealth ($x$), not the growth rate.

$$\dot{W} = \dot{W}_w + \dot{W}_p = (h + \tau)Y - \delta(W_w + W_p) \tag{A3}$$

With the initial condition $W(0) = W_w(0) + W_p(0) = R(0) + K(0)$ (the brackets indicate time periods), equation (A3) is guaranteed that $W(t) = R(t) + K(t)$ also for all times $t > 0$. In other words, total assets (wealth) available in the economy is equal to the total amount of production factors (excluding labour).

Admittedly, this means that there is essentially no, or a very limited, role for financial markets in our model. To the extent that financial markets exist, their only role is to channel savings between profit and wage earners, with the ultimate sole aim to fund the expansion of productive capacity. While here and in most of the main text we assume that there is no rate of return on the ("financial") assets held by wage earners, in a further appendix below, we extend the treatment of financial markets to include a rate of return paid to the holders of financial assets (workers). This changes the steady state value for the distribution of wealth ($x$), but does not affect the growth rate of the economy.

As explained in the main text, our main assumption on autonomous spending is that it depends on wealth, i.e., the variables $W_w$ and $W_p$. Allowing for autonomous consumption by both wage earners and profit earners, we stipulate

$$Z_w = \zeta_w W_w \tag{A4a}$$

and

$$Z_p = \zeta_p W_p \tag{A4b}$$

Note that (A4a) has also been specified in the main text. The $\zeta$ parameters are propensities to consume out of wealth for wage earners and profit earners, respectively. With the variable $x$ defined as the share of $W_w$ in total wealth $(W_w + W_p)$, we immediately have

$$\zeta = \frac{Z}{W} = \zeta_w x + \zeta_p(1 - x) \tag{A5}$$

As specified in the main text, we will assume that $\zeta_w$ is a variable, for which we specify a differential equation. On the other hand, $\zeta_p$ is assumed to be a constant parameter (and the main text assumes $\zeta_p = 0$).

With the equations specified so far, we are able to write a few of the key growth rates in the model:

$$g_W \equiv \frac{\dot{W}_w + \dot{W}_p}{W} = \frac{(1-c_w)\sigma Y - Z_W - \delta W_w + (1-c_p)(1-\sigma)Y - Z_p - \delta W_p}{W} = \zeta\left(\frac{\tau+h}{1-c_w\sigma-c_P(1-\sigma)-\tau-h}\right) - \delta \tag{A6}$$

$$g_Z = \frac{\dot{W}}{W} + \frac{\dot{\zeta}}{\zeta} = \zeta\left(\frac{\tau+h}{1-c_w\sigma-c_P(1-\sigma)-\tau-h}\right) - \delta + \frac{\dot{\zeta}_w x + \zeta_w \dot{x} - \zeta_p \dot{x}}{\zeta_w x + \zeta_p(1-x)} \tag{A7}$$

$$g = g_Z + \frac{\dot{h}}{1-c_w\sigma-c_P(1-\sigma)-\tau-h} \tag{A8}$$

Remember that $g$ is the growth rate of GDP.

Finally, the differential equation for $x$ can be derived as follows:

$$\dot{x} = \frac{\dot{W_w}}{W_w+W_p} - \frac{W_w(\dot{W_w}+\dot{W_p})}{(W_w+W_p)^2} = \frac{(\zeta_w x+\zeta_p(1-x))((1-c_w)\sigma-x(\tau+h))}{1-c_w\sigma-c_p(1-\sigma)-\tau-h} - \zeta_w x \tag{A9}$$

We can also summarize the other five (in addition to A9) differential equations that make up the model with the generalized consumption function (equation numbers refer to the main text, and remember $\Delta = \delta$):

$$\dot{h} = h\gamma(u-\mu) \tag{1}$$

$$\dot{\Phi} = \Phi\frac{u}{v}\left(\frac{\tau}{\Phi} - h\right) \tag{11}$$

$$\dot{\zeta_w} = \iota\zeta_w(\bar{E} - E) \tag{13}$$

$$\dot{u} = u\left(\frac{\big(\zeta_w x+\zeta_p(1-x)\big)(\tau+h)+\dot{h}}{1-c_w\sigma-c_P(1-\sigma)-\tau-h} + \frac{\dot{\zeta_w}x+\zeta_w\dot{x}-\zeta_p\dot{x}}{\zeta_w x+\zeta_p(1-x)} - \frac{hu}{v}\right) \tag{A10}$$

$$\dot{E} = E\left(\frac{\big(\zeta_w x+\zeta_p(1-x)\big)(\tau+h)+\dot{h}}{1-c_w\sigma-c_P(1-\sigma)-\tau-h} + \frac{\dot{\zeta_w}x+\zeta_w\dot{x}-\zeta_p\dot{x}}{\zeta_w x+\zeta_p(1-x)} - \delta - \bar{\rho} - \varphi\Phi\right) \tag{A11}$$

These equations can be solved for the steady state of the model using the procedure outlined in the main text. The steady state expressions presented in the main text are specific for the assumptions $c_P = 0$ and $\zeta_p = 0$. The generalized steady state expressions for $u$, $E$, $\Phi$ and $h$ are identical to the expressions in the main text. The generalized steady state expressions for the two remaining variables are

$$x^* = \frac{v(1+\Phi^*)(\zeta_p+\delta+\bar{\rho}+\varphi\Phi^*)-\mu s_P(1-\sigma)}{v(1+\Phi^*)(\zeta_p+\delta+\bar{\rho}+\varphi\Phi^*)} \tag{A12}$$

$$\zeta_w^* = \frac{[\mu(\sigma s_W+(1-\sigma)s_P)-v(\delta+\bar{\rho}+\varphi\Phi)(1+\Phi^*)](\zeta_p+\delta+\bar{\rho}+\varphi\Phi^*)-\zeta_p\mu s_P(1-\sigma)}{v(1+\Phi^*)(\zeta_p+\delta+\bar{\rho}+\varphi\Phi^*)-\mu s_P(1-\sigma)} \tag{A13}$$

In these expressions, we substituted the definitions $s_P \equiv 1 - c_P$ and $s_w \equiv 1 - c_w$.

## Appendix 2. Introducing a rate of return to workers' assets

As explained in the main text, we model the rate of return on accumulated workers' savings by the introduction of a parameter $r$, which represents the fraction of profits that is paid to workers in return for their savings, which are used by firms (profit earners) to pay for the investments in fixed capital and R&D. With the inclusion $r$ into the model, workers will receive a share $\sigma + rx(1-\sigma)$ of GDP, and profits earners a share $(1-\sigma)(1-rx)$. This changes the equations (equations A2a and A2b) for accumulation of wealth:

$$\dot{W}_w = s_w(\sigma + rx(1-\sigma))Y - \zeta_w W_W - \delta W_w \quad \text{(A2a')}$$

$$\dot{W}_p = s_p(1-\sigma)(1-rx)Y - \zeta_p W_p - \delta W_p \quad \text{(A2b')}$$

It also changes the multiplier, as can be seen in the equation for output:

$$Y = Z_w + Z_p + c_w\big(\sigma + rx(1-\sigma)\big)Y + c_p\big((1-\sigma)(1-rx)\big)Y + (h+\tau)Y \Rightarrow$$

$$Y = \frac{Z_w + Z_p}{1 - c_w\big(\sigma + rx(1-\sigma)\big) - c_p\big((1-\sigma)(1-rx)\big) - (h+\tau)} \quad \text{(3a')}$$

Note that in this equation, we have assumed that the marginal propensity to consume out of current income is unchanged, both for workers and profit earners, even if with $r > 0$, workers' income is partly profits (or returns on savings).

From these basic changes associated to the introduction of $r$, the differential equations of the model can be derived in the same way as before. We find that three equations change, specifically:

$$\dot{u} = u\left(\frac{\big(\zeta_w x + \zeta_p(1-x)\big)(h+\tau) + \dot{h} - \dot{x}(1-\sigma)(s_p - s_w)}{1-(1-s_w)(\sigma + rx(1-\sigma)) - (1-s_p)(1-rx)(1-\sigma) - (h+\tau)} + \frac{\dot{\zeta_w} x + \zeta_w \dot{x} - \zeta_p \dot{x}}{\zeta_w x + \zeta_p(1-x)} - \frac{hu}{v}\right) \quad \text{(A10')}$$

$$\dot{E} = E\left(\frac{\big(\zeta_w x + \zeta_p(1-x)\big)(h+\tau) + \dot{h} - \dot{x}(1-\sigma)(s_p - s_w)}{1-(1-s_w)(\sigma + rx(1-\sigma)) - (1-s_p)(1-rx)(1-\sigma) - (h+\tau)} + \frac{\dot{\zeta_w} x + \zeta_w \dot{x} - \zeta_p \dot{x}}{\zeta_w x + \zeta_p(1-x)} - \delta - \bar{\rho} - \varphi\Phi\right) \quad \text{(A11')}$$

$$\dot{x} = \frac{\big(\zeta_w x + \zeta_p(1-x)\big)[s_w(\sigma + rx(1-\sigma)) - x(h+\tau)]}{1-(1-s_w)(\sigma + rx(1-\sigma)) - (1-s_p)(1-rx)(1-\sigma) - (h+\tau)} - \zeta_w x \quad \text{(A9')}$$

This model (which also includes equations 1, 11 and 13) can be solved for the steady state in essentially the same way as has been done for the case $r = 0$. Related to the fact that in equations (A10') and (A11'), the parameter $r$ only appears in the multiplier (as in A3a'), the steady state solutions for $h$ and $\Phi$ do not change. With equations (1) and (13) unchanged, the steady state solutions for $u$ and $E$ also do not change. Thus, we obtain only steady state expressions for $x^*$ and $\zeta_w^*$ that are different than before, while the other steady state expressions do not change:

$$x^* = \frac{\frac{u}{v}s_p(1-\sigma) - (1+\Phi^*)(\zeta_p + \delta + \bar{\rho} + \varphi\Phi^*)}{\frac{u}{v}s_p(1-\sigma)r - (1+\Phi^*)(\zeta_p + \delta + \bar{\rho} + \varphi\Phi^*)} \quad \text{(A14)}$$

$$\zeta_w^* = \frac{\mu}{v}\frac{(s_w - s_p) r (1-\sigma)}{(1+\Phi^*)} + \frac{\frac{\mu}{v}\left(s_w\sigma + s_p(1-\sigma)\right) - (1+\Phi^*)(\delta + \bar{\rho} + \varphi\Phi^*)}{(1+\Phi^*)}\frac{1}{x^*} - \zeta_p \frac{\frac{\mu}{v}s_p(1-\sigma)(1-r)}{\frac{\mu}{v}s_p(1-\sigma) - (1+\Phi)\left(\zeta_p + \delta + \bar{\rho} + \varphi\Phi\right)} \tag{A15}$$

Especially the expression for $\zeta_w^*$ is rather complicated. However, it can easily be seen that with $r = 0$ it reduces to equation (A13). Similarly, equation (A14) reduces to (A12) for the case $r = 0$. We can also reduce these steady state expressions for the case $r = 1$:

$$x^* = 1 \tag{A16}$$

$$\zeta_w^* = \frac{\mu}{v(1+\Phi^*)}s_w - (\delta + \bar{\rho} + \varphi\Phi^*) \tag{A17}$$

## Appendix 3. Details of the government stabilization model

There are a number of equations that change slightly in the model with a government sector. This appendix presents the details of these equations. First, the introduction of taxes implies that consumption is a function of disposable income:

$$C = Z_H + c(1 - T + (1 - \sigma)D)Y \qquad \text{(2a)}$$

Using also the definitions that are specified in the main text, this leads to the following equation for GDP:

$$Y = (Z_H + Z_G)\frac{1}{1-c(1-T+(1-\sigma)D)-(h+\tau)} \qquad \text{(3b)}$$

Because government bonds are held by private agents (any increase of $G$ will correspond to a private surplus), total assets held by private agents are equal to $W + G$. This modifies the equation for private autonomous spending to

$$Z_h = \zeta_h(W + G) \qquad \text{(A4c)}$$

The equation for government autonomous spending is specified in the main text (A4d), and together these two equations lead to

$$Z = Z_G + Z_h = (\zeta_G + \zeta_h)W + \zeta_h G \qquad \text{(A4e)}$$

In line with our previous derivations (Appendix 1), we also have

$$\frac{\dot{W}}{W} = \frac{(h+\tau)\,[(\zeta_H+\zeta_G)+\zeta_H D]}{1-c(1-t+(1-\sigma)D)-(h+\tau)} - \delta \qquad \text{(A3b)}$$

And from equations (23) and (2a) as well as the definition of $D$, it follows that

$$\frac{\dot{G}}{G} = \frac{1}{D}\left[\zeta_G + \frac{((1-\sigma)D-t)[(\zeta_H+\zeta_G)+\zeta_H D]}{1-c(1-t+(1-\sigma)D)-(h+\tau)}\right] \qquad \text{(A18)}$$

Then it can easily be seen that

$$g_Z = \frac{\dot{\zeta}_g-(\zeta_H+\zeta_G)\delta+\zeta_H\zeta_G}{(\zeta_H+\zeta_G)+\zeta_H D} + \frac{(\zeta_H+\zeta_G)(h+\tau)+\zeta_H((1-\sigma)D-t)}{1-c(1-t+(1-\sigma)D)-(h+\tau)} \qquad \text{(A7a)}$$

With this new equation for $g_Z$, we also have new equations for $\dot{u}$ and $\dot{E}$:

$$\dot{u} = u\left(\frac{\dot{\zeta}_g-(\zeta_H+\zeta_G)\delta+\zeta_H\zeta_G}{(\zeta_H+\zeta_G)+\zeta_H D} + \frac{(\zeta_H+\zeta_G)(h+\tau)+\zeta_H((1-\sigma)D-t)-c\dot{t}+c(1-\sigma)\dot{D}+\dot{h}}{1-c(1-t+(1-\sigma)D)-(h+\tau)} - \frac{hu}{v} + \delta\right) \qquad \text{(7a)}$$

$$\dot{E} = E\left(\frac{\dot{\zeta}_g-(\zeta_H+\zeta_G)\delta+\zeta_H\zeta_G}{(\zeta_H+\zeta_G)+\zeta_H D} + \frac{(\zeta_H+\zeta_G)(h+\tau)+\zeta_H((1-\sigma)D-t)-c\dot{t}+c(1-\sigma)\dot{D}+\dot{h}}{1-c(1-t+(1-\sigma)D)-(h+\tau)} - \bar{\rho} - \varphi\Phi\right) \qquad \text{(8a)}$$